THE UNIVERSITY OF ALBERTA

NUMERICAL INTEGRATION IN n DIMENSIONS

by

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ABSTRACT

A numerical integration procedure which employs m values (ordinates) of the integrand is known as an m-point formula. More generally if the integrand is $w(x^1,\dots,x^n)\ f(x^1,\dots,x^n)$ where $w(\cdot)$ is called the weight function, we could have an m-point formula employing m values of $f(\cdot)$ to approximate the integral over a region in n-space. The m-point formula is said to have precision p if it gives exact results whenever $f(\cdot)$ is a polynomial of degree $\leqq p$. One dimensional integration formulas can be used to obtain a formula over a rectangular region in n-space by forming products of the integrals over each separate variable.

For example, a Gaussian m-point formula in one variable achieves a precision 2m-1. We can obtain from this an m^n-point formula of precision 2m-1 in n dimensions. Although this formula is not unduly extravagant in points for fixed n and large p, one soon has an impractical problem if the number of dimensions n becomes large. A formula economical in points for fixed p and large n is thus of considerable practical value.

The more important known results in this field are as follows:

1) An n + 1 - point formula of precision 2 valid over an arbitrary region in n-space--weight function: arbitrary.

2) An n + 2 - point formula of precision 3 for the n-simplex--weight function: 1.

3) A 2n-point formula of precision 3 for an arbitrary symmetric region is n-space--weight function: symmetric but otherwise arbitrary.

4) A $2n^2 + 1$ - point formula of precision 5 for the n-cube and the n-sphere--weight function: 1.

5) Results on transformations and symmetric regions.

The following results presented in this thesis are believed to be new.

1) A $2n^2 + 1$ - point formula of precision 5 for an arbitrary fully symmetric region in n-space.

2) Two $\frac{4}{3}(n^4 - 5n^3 + 14n^2 - 7n) + 1$ - point formulas of precision 9 valid over an arbitrary fully symmetric region in n-space. For each of these formulas the weight function is fully symmetric but otherwise arbitrary. Results of four particular regions and weight functions are tabulated to 25 dimensions in the Appendix.

These integration formulas were obtained by solving a system of non-linear algebraic equations. Polynomials in one variable orthogonal over the fully symmetric region with respect to the fully symmetric weight function were an aid in obtaining the integration formulas since the zeros of these polynomials can be shown to be the non-linear unknowns in the non-linear algebraic equation.

For large n, the method of attack developed here will in general lead to integration formulas of precision $p = 4k + 1$ (k an integer) using $O(n^{2k})$ points; that is, the formulas are economical for large n and fixed p.

3) A $2(\frac{p+1}{2})^n$ -point formula of arbitrary precision is also developed here for the finite and infinite n-sphere.

4) Estimates of the error committed in performing numerical integration are eminently desirable but not always easily obtained. Towards this end contour integration and asymptotic techniques were employed to extend known results of error bounds for one-dimensional integration to error bounds for repeated Gaussian integration in the case when the integrand is a meromorphic function of n complex variables.

A number of examples are given throughout the thesis and in the Appendix illustrating the accuracy of the formulas developed.

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CHAPTER I

INTRODUCTION

This introductory chapter is intended to facilitate the understanding of the thesis.

Most integrals cannot be evaluated in terms of simple functions. This difficulty is discussed in section 1) and points to the need for numerical integration formulas.

The second section contains (i) some basic definitions of terms used throughout the thesis, and (ii) some examples of numerical integration formulas.

Although the study of numerical integration formulas has received considerable attention, many difficult problems still remain unsolved; this being particularly so for higher dimensions. Section 3) lists the problems this thesis is concerned with.

1) The Purpose of Numerical Integration Formulas

The evaluation of integrals often poses a difficult problem. Although we can evaluate a lot of integrals in terms of simple functions, most of the integrals arising in research or industry cannot be evaluated in terms of simple functions, and we can hope to evaluate them only using a numerical integration formula. This is especially true of multidimensional integrals, due to the multidimensional complexity of the integrand.

For example, in one dimension we have

$$(1) \quad \left\{ \begin{array}{ll} \displaystyle\int_0^x \frac{d\theta}{\sqrt{1 - k^2 \sin^2\theta}} \quad k^2 < 1 , & \displaystyle\int_0^\pi \cos(x \cos\theta) d\theta , \\ \displaystyle\int_0^1 \sin(e^x) dx , & \displaystyle\int_1^a \frac{e^x}{x} dx , \\ \displaystyle\int_\alpha^\beta \frac{dx}{\sqrt{ax^n + bx^{n-1} + \ldots + c}} \quad n \geqq 3 , & \displaystyle\int_\alpha^\beta e^{x^n} dx \quad n \geqq 2 , \end{array} \right.$$

and so on, all of these being integrals that cannot be evaluated in terms of simple functions. Note that we can differentiate every one of the integrands

as often as we please, since continued differentiation does not give rise to new transcendents. Continued integration, however, soon does. Hence if a new integral arises in, say, research, one is lucky to be able to evaluate it in terms of simple functions in one dimension, and if it is multidimensional, he is so much more likely to run into difficulty.

2) Definitions and Examples

i) Definitions. In order to facilitate further discussion, we now make the following definitions.

$$(2) \qquad \begin{aligned} (X) &\equiv (x^1, x^2, \ldots, x^n) \\ dX &\equiv dx^1\, dx^2 \ldots dx^n = \prod_{i=1}^{n} dx^i \ . \end{aligned}$$

That is, X is a $1 \times n$ row vector in euclidean n-space E^n, dX is a hyper-volume element in this space.

We want to find numerical integration formulas of the form

$$(3) \qquad I \equiv \int \cdots \int_{R_n} w(x^1,\ldots,x^n)\, f(x^1,\ldots,x^n)\, dx^1 \ldots dx^n \cong \sum_{j=1}^{m} c_j\, f(x_j^1,\ldots,x_j^n) \ .$$

Here $f(x^1,\ldots,x^n)$ is the continuous function we wish to integrate over a region R_n in E^n with respect to the continuous weight function $w(x^1,\ldots,x^n)$ that does not change sign over the region.

In terms of (2) we may write (3) in the form

$$(4) \qquad I \equiv \int_{R_n} w(X)\, f(X)\, dX \cong \sum_{j=1}^{m} c_j\, f(X_j) \ .$$

Formula (4) is said to be of precision p if it gives exact results whenever $f(X)$ is a polynomial of degree $\leq p$.* That is, the general term of such a polynomial may be written

$$(5) \qquad \prod_{i=1}^{n} (x^i)^{k_i} , \quad \text{where} \quad 0 \leqq \sum_{i=1}^{n} k_i \leqq p , \quad k_i \geqq 0 .$$

When we are looking for integration formulas of the type (4), we have the problem of finding the constants c_j which should preferably be real, and the points X_j which should also be real and lie within the region R_n. Moreover, given p, we want to minimize m.

ii) Examples. Some examples of numerical integration formulas are:

Gauss-Legendre.

$$(6) \qquad \int_{-1}^{1} f(x)\, dx \cong \sum_{j=1}^{m} c_j f(x_j) .$$

Here $w(x) = 1$, $p = 2m-1$, $c_j = 2/\{(1-x_j^2)[P_m'(x_j)]^2\}$, the x_j's are the m zeros of the polynomial $P_m(x)$ orthogonal over the interval $(-1,1)$ with respect to the weight function 1. As an illustration we have

$$(7) \qquad \int_{-1}^{1} f(x)dx \cong \frac{5}{9} f(-\sqrt{\tfrac{3}{5}}) + \frac{8}{9} f(0) + \frac{5}{9} f(\sqrt{\tfrac{3}{5}})$$

where in this case $m = 3$, $p = 5$, $c_1 = c_3 = \frac{5}{9}$, $c_2 = \frac{8}{9}$, $x_1 = -\sqrt{\frac{3}{5}}$, $x_2 = 0$, $x_3 = \sqrt{\frac{3}{5}}$.

Chebychev (one dimension). The following equi-weighted formula is due to Chebychev [25].

$$(8) \qquad \int_{-1}^{1} \frac{f(x)}{\sqrt{1-x^2}}\, dx \cong \frac{\pi}{m} \sum_{j=1}^{m} f(x_j) .$$

---

* We will later show that this definition does not "fit" very well. A discussion of precision is given in Chapter III.

Here $w(x) = 1/\sqrt{1 - x^2}$, $p = 2m - 1$, $c_j = \pi/m$, the x_j's are the m zeros of $T_m(x)$ orthogonal over the interval $(-1,1)$ with respect to the weight function $1/\sqrt{1 - x^2}$. As a more specific example we have

$$(9) \qquad \int_{-1}^{1} \frac{f(x)}{\sqrt{1 - x^2}}\, dx \cong \frac{\pi}{3}\left[f\left(-\sqrt{\tfrac{3}{4}}\right) + f(o) + f\left(\sqrt{\tfrac{3}{4}}\right)\right]$$

where $m = 3$, $p = 5$, $c_1 = c_2 = c_3 = \frac{\pi}{3}$, $x_1 = -\sqrt{\frac{3}{4}}$, $x_2 = 0$, $x_3 = \sqrt{\frac{3}{4}}$.

Tyler's formula for the n-cube.

$$(10) \qquad \int_{-1}^{1} \cdots \int_{-1}^{1} f(x^1,\ldots,x^n)dx^1 \ldots dx^n = \frac{2^{n-1}}{n} \sum_{j=1}^{2n} f(x_j^1,\ldots,x_j^n) \quad .$$

Here $w(X) = 1$, $m = 2n$, $p = 3$, $c_j = 2^{n-1}/n$, $x_j^i = \sqrt{\frac{n}{3}}\,(\delta_{j,2i-1} - \delta_{j,2i})$, $\delta_{i,j}$ being the well-known Kronecker delta.

3) The Problem.

One way of obtaining a numerical integration formula over a rectangular region in n dimensions is by taking products of one dimensional integrals over each separate variable. Proceeding thus, we would find the number of integration points increasing exponentially in n. For example, by taking products of formulas of the type (6) and/or (8) and keeping the precision $p = 2m-1$ in each variable, we would require m^n points over a rectangular region in n dimensions. As the number of dimensions increases we would soon find ourselves faced with a problem that takes too long to evaluate even with the fastest digital computer. Hence there is a need for numerical integration formulas that require fewer points for a large number of dimensions n.

Numerical integration formulas of precision p in n dimensions requiring fewer than $(\frac{p+1}{2})^n$ points have been found by earlier workers in

the field but only up to precision three in the general case and up to precision five in two particular cases. Such a limited class of formulas severely restricts the accuracy obtainable when performing numerical integration.

Although $(\frac{p+1}{2})^n$ - point formulas require an extremely large number of points when the dimension n is large, we cannot overlook the superior accuracy of these formulas for a large class of functions that have all their derivatives continuous. Moreover, these are presently the only available formulas of arbitrary precision. Of the many different regions possible in higher dimensions, we have seen only such formulas (of arbitrary precision) for rectangular regions in n-space.

Very little work has been done regarding error estimates of integration formulas in n dimensions. This may be expected since the topic is relatively new. Most workers have concentrated on the task of finding workable results and in general the results known are of such complexity that practical error evaluation would be difficult to obtain. Apriori error estimates are preferable when they can be found, particularly if the time required to evaluate these is small compared with the time required to evaluate the approximate integral.

CHAPTER II

REVIEW OF THE LITERATURE

This chapter of the thesis states the known results of the field. It is by no means a complete summary of approximate integration, but consists rather of known results of n-dimensional approximate integration pertaining to the problems discussed in Chapter I. Stroud [24] gives a very extensive bibliography of approximate integration up to 1960.

In section 1) we state results of nineteenth century authors, C. F. Gauss, P. L. Chebychev and J. C. Maxwell.

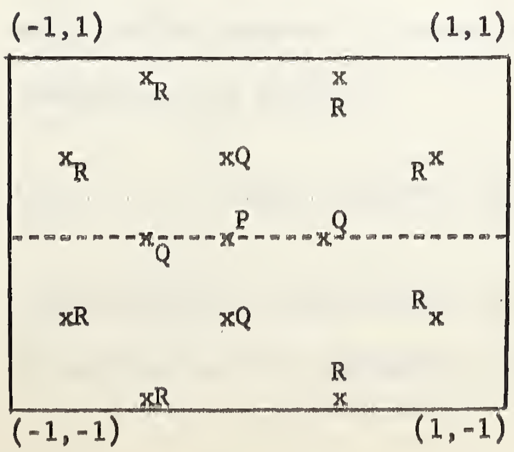
The principal results of section 2) are concerned with (a) extending integration formulas known over one region to other regions, (b) construction of integration formulas for basic regions such as n-cubes, n-simplexes, n-spheres, (c) the construction of integration formulas for arbitrary regions in n-space, (d) numerical integration by Monte Carlo Method, and (e) a derivative error bound of von Mises.

1) Nineteenth Century Results

Gauss [1] was the first to solve the problem of obtaining maximum possible accuracy with m points in one dimension. By chosing his m points x_j to be the m zeros of $P_m(x)$, he was able to obtain precision $p = 2m - 1$.

J. C. Maxwell [2], in 1877, found numerical integration formulas for the rectangle and for the 3-dimensional parallelelepipedon. For the rectangle, he obtained a formula of precision 7 using 13 points:

$$(1) \quad \int_{-1}^{1}\int_{-1}^{1} f(x,y)dx\, dx \cong P\, f(0,0) + Q\sum f(\pm p,0) + f(0,\pm p) + R\sum f(\pm q,\pm r) \text{ .}$$



In order to set up equations and solve for the points p, q, r and the numbers P, Q, R are shown in the figure on the left, he first transformed the rectangle into the square.

He then found the solution to the following six simultaneous non-linear algebraic equations.

$$
(2)\quad \begin{aligned}
P + 4Q + 8R &= 4 \\
Qp^2+2R(q^2+r^2) &= 2/3 \\
Qp^4+2R(q^4+r^4) &= 2/5 \\
Qp^6+2R(q^6+r^6) &= 2/7 \\
2R\, q^2\, r^2 &= 1/9 \\
Rq^2r^2(q^2+r^2) &= 1/15
\end{aligned}
$$

The solutions he obtained are

$$
(3)\quad \begin{aligned}
p^2 &= 12/35 & \qquad P &= 8/162 \\
q^2 &= 3/5\,[1+(6/31)^{\frac{1}{2}}] & Q &= 98/162 \\
r^2 &= 3/5\,[1-(6/31)^{\frac{1}{2}}] & R &= 31/162
\end{aligned}
$$

For the three dimensional case of precision 7 he obtained two 27 point solutions, each of which had points that lay outside the region of integration. This sort of thing occurs quite frequently when one tries to find integration formulas in more than two or three dimensions (see for example, equation (10) of Chapter I for $n > 3$). By a different selection of points, Maxwell could have avoided obtaining points that lie outside the region of integration (see M.T.A.C. v. 12 P. 274). Whenever possible we want to avoid obtaining formulas with such points since, for example, if we wanted to compute the weight of a petroleum product by measuring its density which varied throughout a rectangular tank, the formulas command us to measure the density outside of the tank.

For the same reason, we want to try to avoid obtaining formulas with complex points. An example of this phenomenon is furnished by the Chebychev type formula:

$$
(4)\qquad \int_{-1}^{1} w(x)\, f(x)\, dx \cong \frac{\alpha}{m} \sum_{j=1}^{m} f(x_j)\ ; \qquad \alpha = \int_{-1}^{1} w(x)\, dx\ .
$$

(An integration formula such as this, for which all the c_j's are equal, is said to be of a Chebychev type, in honor of Chebychev who first tried

to find such integration formulas). For the integration formula (14) to have precision of at least m for the particular case $w(x) = 1$, the x_j's can be shown (see [3]) to be the m zeros of the polynomial part of

$$(5) \qquad \exp\left[\frac{m}{2} \int_{-1}^{1} \log(x - t)dt \right].$$

We will call the polynomial part of (5) $G_m(x)$. Chebychev found that all the polynomials $G_1, G_2, \ldots, G_7$ and G_9 had real zeros within the interval $(-1,1)$. G_8, however, had six complex zeros. Chebychev stopped here, leaving the question as to what the zeros of $G_m(x)$ are like for $m \geqq 10$ open. Later, after Chebychev's death it was shown [4] that for $m \geqq 10$ every $G_m(x)$ had at least one pair of complex roots.

2) Recent Developments

The problem of obtaining numerical integration formulas in higher dimensions received little attention until the last 20 years. Indeed no practical use could be found for these until the development of electronic computers (around 1947) made it possible to envisage extensive computations of integrals in more than one variable. Since then new attempts at finding numerical integration formulas in higher dimensions have been made.

The following definitions facilitate the presentation of the results of this section.

Definitions

(a) A set R_n in E^n is said to be fully symmetric if $X \in R_n$ implies $Y \in R_n$ where Y is any point obtainable from X by permutations and by changes of sign of the coordinates of X.

(b) A function g defined in a fully symmetric set is fully symmetric if $g(X) = g(Y)$.

(c) A numerical integration formula is said to be fully symmetric if the sets of evaluation points $X_1, X_2, \ldots, X_m$ are decomposable into fully symmetric sets and if the weight function $w(X)$ is a fully symmetric function.

Tyler [5] in 1953 found integration formulas over regions bounded by parabolas and the hypercube. His numerical integration formula for the hypercube is of precision 3, very similar to formula (10) of Chapter I, and requires $2n + 1$ points.

Hammer, Marlowe, Stroud and Wymore (but mainly Hammer and Stroud) have written extensively on the problem of finding numerical integration formulas in higher dimensions. They have given numerical integration formulas up to precision 3 for the n-simplex [6], up to precision 5 for the n-cube and the n-sphere [7]; they have given several theorems for extending available results into higher dimensions and into regions related to given regions by transformations. We present here two important theorems due to Hammer and Wymore [8].

i) Transformation theorem

Suppose we have an integration formula which may be written in terms of its error

$$(6) \qquad E(f,R_n) = \sum_{j=1}^{m} a_j\, f(X_j) - \int_{R_n} w_1(X)\, f(X)\, dX$$

where w_1 and f are continuous functions of X in a region R_n of euclidean n-space E^n.

Let S_n be another region in E^n such that there exists a transformation

$$(7) \qquad Y = Y(X) \qquad (= y^1(X), \ldots, y^n(X))$$

with a continuous non-vanishing jacobian

$$(8) \qquad J = J(X) = \begin{vmatrix} \frac{\partial y^1}{\partial x^1} & \cdots\cdots & \frac{\partial y^n}{\partial x^1} \\ \vdots & & \\ \frac{\partial y^1}{\partial x^n} & \cdots\cdots & \frac{\partial y^n}{\partial x^n} \end{vmatrix}$$

which maps R_n onto S_n .

Further, if $w_1(X) = w_2(Y)$ and $g(Y)$ is a continuous function of Y in S_n, we have

$$(9) \qquad \int_{S_n} w_2(Y)\, g(Y)\, dY = \int_{R_n} w_2(Y)\, g(Y)\, |J(X)|\, dX$$
$$= \int_{R_n} w_1(X)\, h(X)\, |J(X)|\, dX$$

where we have let $g(Y) = h(X)$.

Hence, by formula (6) we have

$$(10) \qquad E(|J|h,R_n) = \sum_{j=1}^{m} a_j |J(X_j)| - \int_{S_n} w_2(Y)\, g(Y)\, dY \; .$$

By formulas (10) and (6) we thus have the following

THEOREM:

$$(11) \qquad E(g,S_n) = \sum_{j=1}^{m} b_j\, g(Y_j) - \int_{S_n} w_2(Y)\, g(Y)\, dY$$

where $b_j = a_j|J(X_j)|$, $Y_j = Y(X_j)$ and hence for every formula of the type (6) in R_n there is a corresponding formula of the same type in S_n with error function $E(g,S_n) = E(|J|h,R_n)$.

COROLLARY: If $Y(X)$ is a non-singular transformation for which $J(X)$ is a constant, then $E(g,S_n) = E(|J|h,R_n) = |J|\, E(h,R_n)$ and hence if formula (6) has precision p in R_n, then formula (11) has precision p in S_n. In particular, if $Y(X)$ is an affine transformation, $J(X)$ is a constant.

ii) Cartesian product theorem

Let R be the Cartesian product of two regions R_1 and R_2 in lower dimensional euclidean spaces. Let us designate X in R_1, Y in R_2, so that every Z in R can be written (X,Y). Suppose we have numerical integration formulas in R_1 and R_2 with weight functions $w_1(X)$ and $w_2(Y)$ respectively. Together with their error functions, these may be written

$$(12) \qquad E_1(R_1,f_1) = \sum_{i=1}^{m_1} a_i\, f_1(X_i) - \int_{R_1} w_1(X)\, f_1(X)\, dX$$

$$(13) \qquad E_2(R_2,f_2) = \sum_{j=1}^{m_2} b_j\, f_2(Y_j) - \int_{R_2} w_2(Y)\, f_2(Y)\, dY \quad .$$

Using (12) and (13) we obtain the following

THEOREM: Let $f(Z) = f(X,Y)$ be defined and continuous on the region $R = R_1 \times R_2$. Then, if we define

$$(14) \qquad E(R,f) = \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} a_i\, b_j\, f(X_i,Y_j) - \int_{R_2} \int_{R_1} w_1(X)\, w_2(Y)\, f(X,Y)\, dX\, dY$$

we have

$$(15) \qquad E(R,f) = \sum_{j=1}^{m_2} b_j\, E_1(R_1,f(X,Y_j)) + \int_{R_1} w_1(X)\, E_2(R_2,f(X,Y))\, dX$$

$$= \sum_{i=1}^{m_1} a_i\, E_2(R_2,f(X_i,Y)) + \int_{R_2} w_2(Y)\, E_1(R_1,f(X,Y))\, dY \; .$$

COROLLARY (1): If $f(X,Y)$ is a function such that $E_1(R_1,f(X_i,Y))=0$ for each $Y \in R_2$ and $E_2(R_2,f(X,Y_j))=0$ for each $X \in R_1$ then $E(R,f) = 0$ and formula (14) is exact for $f(X,Y)$.

Now, if F_1 is a class of functions defined on R_1 and F_2 is a class of functions defined on R_2 then the Cartesian product class $F = F_1 \times F_2$ of functions defined on $R_1 \times R_2$ is the class of all functions $f(X,Y)$ such that $f(X,Y) \in F_1$ for each Y in R_2 and $f(X,Y) \in F_2$ for each $X \in R_1$.

COROLLARY (2): If classes of functions F and G respectively defined on R_1 and R_2 are representable as all linear combinations of basis sets of functions $f_1,\ldots,f_r$, $g_1,\ldots,g_s$ respectively then $(F \times G)$ is the set of all functions with $f_i\ g_j$ as basis.

PROOF (of corollary (2)):

If $h = \sum_{i=1}^{r} \sum_{j=1}^{s} c_{ij}\ f_i\ g_j$ where the c_{ij}'s are constants, then $h \in F \times G$. To show that $h \in F \times G$ implies h is a linear combination of the $(f_i g_j)$, we observe that $h = \sum_{i=1}^{r} a_i f_i = \sum_{j=1}^{s} b_j g_j$ where the a_i are unique functions on R_2 and the b_j are unique functions on R_1.

Let $X_1,\ldots,X_s$ be the points in R_1 such that the determinant $|f_i(X_k)| \neq 0$. Then we have with b_{jk} the value of b_j at X_k, $\sum_{i=1}^{r} a_i f_i(X_k) = \sum_{j=1}^{s} b_{jk}\ g_j$, $k = 1,\ldots,s$, which are identities on R_2. From this set of equations we can solve for $a_i = \sum_{j=1}^{s} c_{ij}\ g_j$, the c_{ij}'s being unique since the determinant $|f_i(X_k)| \neq 0$.

iii) <u>Implications of the theorems</u>

What are the implications of the above theorems? The transformation theorem, tells us that, for example, if we have a numerical integration

formula of precision p over a given region, we automatically have an integration formula of the same precision over a region related to the given one by an affine transformation. This result is not only useful in extending known integration formulas to regions related to a given one by affine transformations, but it is also a tremendous aid in obtaining new integration formulas.

To illustrate this, the number of different monomials in the polynomial

$$(1 + x^1 + x^2 + \dots + x^n)^p$$

is equal to the number of ways of assigning p indistinguishable objects to $(n + 1)$ different groups, which is well known to be the binomial coefficient,

$$\binom{n+p}{p} = \frac{(n+1)_p}{p!} = \frac{(p+1)_n}{n!} = \binom{n+p}{n} .$$

Hence finding an integration formula of precision p over an arbitrary region in E^n would require the solution of

$$\frac{(n+1)_p}{p!}$$

simultaneous non-linear algebraic equations. This number of equations not only becomes very large as n or p increases, but it is dependent on n and p.

If, however, we restrict ourselves to a fully symmetric region when looking for integration formulas, relying on the above transformation theorem to extend our results to other regions, the total number of simultaneous non-linear algebraic equations becomes independent of n, and considerably less. For example, to obtain an integration formula of precision

7 over arbitrary 3-space would require the solution of 120 simultaneous non-linear algebraic equations. For a fully symmetric region however, (in 3-space as in n-space) the total number of simultaneous non-linear algebraic equations reduces to 7.

The second theorem tells us that if we have an integration formula over the circle and another over an interval along a line, we immediately have one for the cylinder; other examples could easily be adduced. To obtain an integration formula over the square, we would simply use the fact that the square is a Cartesian product of two lines of the same lenth, together with a Gauss-Legendre integration formula along a line.

Note (1): If we need m_1 points to obtain precision p_1 in the variables X in R_1 and m_2 points to obtain precision p_2 in the variables Y in R_2, then we need $m_1 \times m_2$ points to obtain precision p_1 in the variables X and precision p_2 in the variables Y in the region $R_1 \times R_2$. That is, the precision we can guarantee in our Cartesian product integration formula is the minimum of p_1 and p_2.

iv) A result for cones

We now develop an interesting integration formula for cones, due to Hammer, Marlowe and Stroud [6].

Let R_n be an n-dimensional region on the hyperplane $y = 1$ which is embedded in E^{n+1}. We represent the points in E^{n+1} by (X,y), X being a point in E^n. Then the set of all points yR_n, where $0 \leqq y \leqq 1$ is a cone C_{n+1} with base R_n and vertex at the origin in E^{n+1}. Let $f(X,y)$ be a function defined over C_{n+1} and suppose we have a numerical integration formula over the base R_n of C_{n+1}. If, for example

$$(16) \qquad \int_{R_n} f(X,1)\, dX \cong \sum_{i=1}^{m_1} a_i\, f(X_i,1)$$

then

$$(17) \qquad \int_{C_{n+1}} f(X,y)\, dX\, dy = \int_0^1 \int_{yR_n} f(X,y)\, dX\, dy$$

$$\cong \int_0^1 y^n \sum_{i=1}^{m_1} a_i\, f(X_i y, y)\, dy$$

since the jacobian of the affine transformation from R_n to yR_n is y^{-n}. If we now define

$$(18) \qquad g(y) \equiv \sum_{i=1}^{m_1} a_i\, f(X_i y, y) \quad ,$$

we have

$$(19) \qquad \int_{C_{n+1}} f(X,y)\, dX\, dy \cong \int_0^1 y^n\, g(y)\, dy \quad .$$

Since numerical integration formulas of the form

$$(20) \qquad \int_0^1 y^n\, g(y)\, dy \cong \sum_{j=1}^{m_2} b_j\, g(y_j)$$

certainly exist, we obtain the result

$$(21) \qquad \int_{C_{n+1}} f(X,y)\, dX\, dy \cong \sum_{i=1}^{m_1} \sum_{j=1}^{m_2} a_i\, b_j\, f(X_i y_j, y_j) \quad .$$

Hence if (16) holds precisely for polynomial $f(X)$ of n variables of at most degree p over a region R_n, and if formula (20) holds precisely for polynomials $g(y)$ of at most degree p in y, then (21) holds over the cone C_{n+1} for all polynomials containing monomials of at most degree p in its $n + 1$ variables.

v) A result for simplexes

Let the vertices of the n-simplex, S_n be $Y^n_1, \ldots, Y^n_{n+1}$, and then its centroid is given by

(22) $$C^n = \frac{1}{n+1} \sum_{j=1}^{n+1} Y^n_j \quad .$$

Let V_n be the hypervolume of S_n. We employ the superscript n in our notation since we later want to discuss S_n and S_{n-1} simultaneously.

THEOREM. An integration formula exact for the general cubic polynomial over S_n for $n \geqq 1$ is given by

(23) $$\int_{S_n} f(X^n)\, dX^n \cong a_n \sum_{j=1}^{n+1} f(X^n_j) + b_n\, f(C^n)$$

where

(24) $$a_n = \frac{(n+3)^2}{4(n+1)(n+2)} V_n \,, \quad b_n = -\frac{(n+1)^2}{4(n+2)} V_n$$

$$X^n_j = \frac{2}{n+3} Y^n_j + \frac{n+1}{n+3} C^n \,; \quad j = 1 \text{ to } n+1 \;.$$

(Before beginning the proof we remark that the points X_j are on the median lines of S_n and the statement of the theorem is in symmetric form).

PROOF. There exists an affine transformation mapping any simplex onto any other. Hence to prove the theorem we may use the vertices

$$Y^n_1 = (0,\ldots,0) \,, \qquad Y^n_2 = (1,0,\ldots,0)$$

$$Y^n_3 = (1,1,0,\ldots,0),\ldots, \qquad Y^n_{n+1} = (1,0,\ldots,0,1)$$

It is readily verified that the formula given holds for $n = 1$ and $n = 2$. Hence we assume that it holds for E^{n-1} and proceed to show that it also holds for E^n, where $n - 1 \geqq 2$. Let $f(x^1,x^2,\ldots,x^n) = f(X^n)$ be a cubic

polynomial in $x^1, x^2, \ldots, x^n$. Then $f(1, x^2, \ldots, x^n)$ is a cubic polynomial in $x^2, \ldots, x^n$. Now using a result established for cones (equation (17)) we may write

$$(25) \qquad \int_{S_n} f(X^n)\, dX^n = \int_0^1 \left(\int_{x^1 S_{n-1}} f(X^n)\, dX^{n-1} \right) dx^1$$

$$= \int_0^1 (x^1)^{n-1} \left[a_{n-1} \sum_{j=2}^{n+1} f(x^1 X_j^{n-1}) + b_{n-1}\, f(x^1 C^{n-1}) \right] dx^1$$

where S_{n-1} is the $(n-1)$ - simplex with vertices $Y_2^{n-1}, \ldots Y_{n+1}^{n-1}$ $(Y_j^{n-1} = Y_j^n)$ in the hyperplane $x^1 = 1$, $C^{n-1} = \frac{1}{n} \sum_{j=2}^{n+1} Y_j^{n-1}$, where X_j^{n-1}, a_{n-1} and b_{n-1} are given by equation (24), with n replaced by $n - 1$, j running from 2 to $n + 1$. It is observed that the hypervolume V_{n-1} is $1/(n-1)!$ and V_n is $1/n!$. Let $f(X^n)$ be the monomial $(x^1)^{k_1} (x^2)^{k_2} (x^3)^{k_3}$ where $0 \leqq k_1 + k_2 + k_3 \leqq 3$. Using (25) we find on substitution and simplification that

$$(26) \qquad \int_{S_n} (x^1)^{k_1} (x^2)^{k_2} (x^3)^{k_3}\, dX^n = \frac{V_n[(n+2)^{2-k_2-k_3}(3^{k_2}+3^{k_3}+n-2)-n^{3-k_2-k_3}]}{4(n+1)\,(n + k_1 + k_2 + k_3)}$$

On the basis of our assumption (26) gives the value of the integral indicated. On the other hand, formula (23) (and (24)) applied to $f(X^n) = (x^1)^{k_1} (x^2)^{k_2} (x^3)^{k_3}$ gives

$$(27) \qquad \frac{V_n}{4(n+1)(n+2)} \left\{ (n+3)^{2-k_1-k_2-k_3} [n^{k_1} + (n+2)^{k_1} (3^{k_2} + 3^{k_3} + n-2)] - (n+1)^{3-k_1-k_2-k_3} n^{k_1} \right\}$$

Now it may then be directly verified that (27) gives the same result as (26) for $0 \leqq k_1+k_2+k_3 \leqq 3$, $0 \leqq k_1 \leqq 3$, $k_2 \geqq k_3$.

Hence in view of the symmetry with which the last $n-1$ coordinates appear in the set of vertices $Y_1, \ldots, Y_{p+1}$, (4) is verified as correct for all monomials of the form $(x^1)^{k_1} (x^k)^{k_j} (x^\ell)^{k_\ell}$ for $0 \leqq k_1 + k_j + k_\ell \leqq 3$, $j \neq \ell$ where j and ℓ are taken from 2 to n. The only monomial type thus omitted is $x^2 x^3 x^4$ provided $n \geqq 4$. Using formula (25) for this monomial we find

$$(28) \qquad \int_{S_n} x^2 x^3 x^4 \, dX^n = \frac{V_n}{4(n+1)(n+2)(n+3)}$$

which coincides with the value obtained on substituting $f(X^n) = x^2 x^3 x^4$ in (23) (and (24)). Hence our induction is complete and formulas (23) and (24) hold. (The result (23) and (24) and proof is due to P. C. Hammer and A. H. Stroud, published in M.T.A.C. V.10, p.137).

This is then, a useful result for polyhedrons in that every polyhedron is made up of a finite number of n-simplexes. However, since, for example the n-cube, is made up of $n!$ n-simplexes, it is, in general, worthwhile using a different integration formula for symmetric polyhedrons.

vi) Thacher's matrix method of attack

It was noted earlier that in order to find an integration formula of the form (4) of precision p over an arbitrary region R_n in E^n we would need to solve $(n+1)_p/p!$ simultaneous non-linear algebraic equations. For $w(X) = 1$, these equations may be written in the explicit form

$$(29) \qquad I_{k_1,\ldots,k_n} \cong \int \cdots \int_{R_n} \prod_{i=1}^{n} (x^i)^{k_i} \, dx^i = \sum_{j=1}^{m} c_j \prod_{i=1}^{n} (x_j^i)^{k_i}$$

where the m c_j's and the mn x_j's are unknowns, $0 \leqq \sum_{i=1}^{n} k_i \leqq p$. For a symmetric region the number of equations will of course, be considerably

less since we need make no distinction between the variables x^i, x^j $(i,j=1,\ldots,n)$ of a symmetric region.

Define a set of $m \times m$ diagonal matrixes by

$$G = [C_j \, \delta_{hj}] \tag{30}$$

$$Y^i = [x_j^i \, \delta_{hj}] \quad . \tag{31}$$

In terms of these matrixes (29) becomes

$$\operatorname{tr} \left\{ G \prod_{i=1}^{n} (Y^i)^{k_i} G \right\} = I_{k_1,\ldots,k_n} \quad . \tag{32}$$

In particular, for hypercubes of edge length 2.

$$I_{k_1,\ldots,k_n} = \begin{cases} 0 & \text{if at least one } k_i \text{ is odd} \\ \prod_{i=1}^{n} \left(\frac{2}{k_i+1}\right) & \text{if no } k_j \text{ is odd.} \end{cases}$$

For a second degree formula we have

$$\operatorname{tr} \{G\,G\} = 2^n \tag{33}$$

$$\operatorname{tr} \{GY^iG\} = 0 \tag{34}$$

$$\operatorname{tr} \{GY^iY^jG\} = \frac{2^n}{3} \delta_{ij} \quad . \tag{35}$$

These equations in the traces may be converted to vector equations if we introduce the m-dimensional column vector, ϵ, with all elements unity, and its transpose ϵ^T . Then, if we define the vectors ζ, ζ_i, ζ_{ij}, ..., by $\zeta = G\epsilon$, $\zeta_i = Y^iG\epsilon$, $\zeta_{ij} = Y^iY^jG\epsilon$, and so on, (33), (34) and (35) become

$$\epsilon^T \, G \, G \, \epsilon = (G \, \epsilon)^T \, (G\epsilon) = \zeta^T\zeta = 2^n \tag{36}$$

(37) $\epsilon^T G Y^i G \epsilon = (G \epsilon)^T (Y^i G \epsilon) = \zeta^T \zeta_i = 0$

(38) $\epsilon^T G Y^i Y^j G \epsilon = (Y^i G\epsilon)^T (Y^j G\epsilon) = \zeta_i^T \zeta_j = (Y^i Y^j G\epsilon)^T (G\epsilon) = \zeta_{ij}^T \zeta = \frac{2^n}{3} \delta_{ij}$

These, however, are recognized merely as orthogonality relations among $(n+1)$ vectors $\zeta, \zeta_1, \ldots, \zeta_n$ and the normalization requirement that $|\zeta|^2 = 2^n$, $|\zeta_i|^2 = 2^n/3$. Now $(n+1)$ orthogonal vectors span a vector space of dimension $(n+1)$ and this space must be a subspace of the vector space of dimension m consisting of all m-dimensional vectors. Thus a second degree integration formula can be obtained with $m = n+1$ and for any higher value of m.

The above argument has also furnished an explicit algorithm for constructing examples of such formulas by orthogonalizing any linearly independent set of $(n+1)(n+1)$ dimensional vectors, and applying proper normalization conditions.

For example, orthogonalizing the set $(1,1,1)$, $(3, -\sqrt{3} \tan \theta, \sqrt{3} \tan \theta)$, $(3\sqrt{3} \tan \theta, 0, 0)$ results in

(39) $\zeta = (2/\sqrt{3}, 2/\sqrt{3}, 2/\sqrt{3})$

(40) $\zeta_1 = (\frac{2\sqrt{2}}{3} \cos \theta, \frac{2\sqrt{2}}{3} \cos(\theta + \frac{2\pi}{3}), \frac{2\sqrt{2}}{3} \cos(\theta + \frac{4\pi}{3}))$

(41) $\zeta_2 = (\frac{2\sqrt{2}}{3} \sin \theta, \frac{2\sqrt{2}}{3} \sin(\theta + \frac{2\pi}{3}), \frac{2\sqrt{2}}{3} \sin(\theta + \frac{4\pi}{3}))$.

Hence, for this case, we obtain the integration formula

(42)
$$\int_{-1}^{1} \int_{-1}^{1} f(x,y)\, dx\, dy \cong \frac{4}{3} \Big\{ f(\frac{2}{\sqrt{3}} \cos \theta, \frac{2}{\sqrt{3}} \sin \theta) + f(\frac{2}{\sqrt{3}} \cos(\theta + \frac{2\pi}{3}), \sin(\theta + \frac{2\pi}{3}) + f(\frac{2}{\sqrt{3}} \cos(\theta + \frac{4\pi}{3}), \frac{2}{\sqrt{3}} \sin(\theta + \frac{4\pi}{3})) \Big\}$$

which is exact whenever $f(x,y)$ is a polynomial of degree two or less.

For 3rd degree formulas we have, along with (36), (37) and (38) the condition

$$(43)\qquad \epsilon^T G Y^i Y^j Y^k G \epsilon = (Y^i Y^j G \epsilon)^T (Y^k G \epsilon) = \zeta_{ij}^T \zeta_R = 0 \quad .$$

Thus we must consider the $n(n+1)/2$ new vectors ζ_{ij} in addition to ζ and ζ_i. These new vectors fall into two classes: the n ζ_{ii} are orthogonal to every ζ_k but not to ζ, while the $n(n-1)/2$ ζ_{ij} $(i \neq j)$ are orthogonal to both sets, or else are null vectors.

Let us consider first the case where all the ζ_{jk} are null vectors. This implies that unless one or more elements of ζ is zero, in which case the basic integration formula includes redundant points with zero weight, only one of the Y^i can have any given element different from zero. The ζ_{ii} cannot be null vectors in view of (38) while from (37) unless the C_j differ in sign, Y^i must include elements of both signs, and thus at least two non-zero elements. We therefore conclude that the dimension m of Y^i must be at least $2n$.

For an equally weighted formula G is a scalar matrix which we may write as gE where E is the identity matrix. From (33) for a $2n$ point formula, g^2 must have the value $2^{n-1}/n$. For the minimum number of points Y^i will have but 2 non-zero elements of opposite sign and from (34) of equal magnitude, and these may be arranged in order such that

$$(44)\qquad (Y^i)_{hj} = x^i(\delta_{hj}\, [\delta_{j,2i-1} - \delta_{j,2i}]) \;,$$

a diagonal matrix with the $(2i-1)$'th element equal to x^i and the $(2i)$'th to $-x^i$. From (35) it follows that

(45) $$2(x^i)^2 g^2 = 2n/3$$

so that for all i

(46) $$x^i = \sqrt{\frac{n}{3}}$$

and we have the family of $2n$ point third degree integration formulas given in equation (10).

If one or more of the ζ_{ij} is non-null there must be $n+2$ or more orthogonal vectors, and the minimum value of m must be at least $n+2$. This lower bound cannot always be achieved, the only region for which we have seen an $(n+2)$-point third degree formula being the n-simplex (formula (23)). Thacher claims, moreover, that it can be shown that no 5 point third degree formula exists for the cube.

This method of obtaining integration formulas, due to H. Thacher Jr. is interesting in that he outlines a different attack by which he first finds the minimum number of points required and then proceeds to find the points and weights (V.11 p.189 M.T.A.C.). For higher precisions than 3 the method becomes exceedingly complicated, since additional non-linearities are introduced.

vii) <u>A remark on the uniqueness of evaluation points</u>

Note that for $n > 3$ in formula (10) of Chapter I, the points x^i_j lie outside the n-cube. Since the distance from the center of the n-cube to its furthest edge is $\sqrt{n}$, $\sqrt{n} > \sqrt{\frac{n}{3}}$, so that for $n > 3$ we can rotate the points of integration back into the n-cube, provided a rotation of the points does not reduce the precision of our integration formula.

Now, a rotation is a particular affine transformation for which the elements of the matrix $[\alpha_{k\ell}]$ of the transformation satisfy

(47) $$\sum_{i=1}^{n} \alpha_{ik}\,\alpha_{i\ell} = \delta_{k\ell}$$

When integrating the monomial $\bar{x}^k\,\bar{x}^\ell$ of degree 2 over the n-cube, we obtain

(48) $$\int_{-1}^{1}\cdots\int_{-1}^{1} \bar{x}^k\,\bar{x}^\ell\, d\bar{x}^1 \ldots d\bar{x}^n = \frac{2^n}{3}\,\delta_{k\ell}$$

Applying the rotation $\bar{x}^k = \sum_{i=1}^{n} \alpha_{ik}\,x^i$ and integrating the transformed monomial over the same n-cube we obtain

(49) $$\int_{-1}^{1}\cdots\int_{-1}^{1} \left(\sum_{i=1}^{n}\sum_{j=1}^{n} \alpha_{ik}\,\alpha_{j\ell}\,x^i\,x^j\right) dx^1 \ldots dx^n$$

$$= \int_{-1}^{1}\cdots\int_{-1}^{1} \left[\left(\sum_{i=1}^{n} \alpha_{ik}\,\alpha_{i\ell}\,(x^i)^2\right) dx^1 \ldots dx^n\right.$$

$$\left. + \left(\sum_{i\neq j} \alpha_{ik}\,\alpha_{j\ell}\,x^i\,x^j\right) dx^1 \ldots dx^n\right]$$

$$= \frac{2^n}{3}\,\delta_{k\ell}$$

Hence we have shown another way of justifying a result for the n-cube due to A. H. Stroud (V.11 p.257, M.T.A.C.), if $X_j = (x_j^1,\ldots,x_j^n)$ then

(50) $$\begin{cases} x_j^{2i-1} = \frac{2}{3}\cos\frac{2ij\pi}{n+1} & i = 1, \ldots, [\frac{1}{2}n] \; . \\ x_j^{2i} = \frac{2}{3}\sin\frac{2ij\pi}{n+1} & (\text{If } n \text{ is odd, } x_j^n = (-1)^j/\sqrt{3}) \end{cases}$$

Moreover we have shown that for the n-cube the points of integration of the second and third degree integration formulas are not uniquely determined.

viii) A precision 2 formula over an arbitrary region in E^n

A. H. Stroud (V.14, p.21, M.T.A.C.) extended H. Thacher's matrix attack to obtain an n+1 - point formula of precision 2 valid over an arbitrary region in E^n that (together with the arbitrary weight function) satisfies certain conditions of non-degeneracy. We summarize his procedure in what follows.

Assume the integration formula

$$\int_{R_n} w(X)\ f(X)\ dX \cong \sum_{j=0}^{n} c_j\ f(X_j)$$

to be of precision 2 and to require n+1 points. To establish our notation at this point we define

$$a_{oo} = \int_{R_n} w(X)\ dX$$

(51)

$$a_{oi} = \int_{R_n} w(X)\ x^i\ dX \qquad a_{k\ell} = \int_{R_n} w(X)\ x^k\ x^\ell\ dX$$

To obtain the above integration formula we want to find the numbers c_j and the points v_j^i given in the equations

$$(52)\quad \begin{cases} \displaystyle\sum_{j=0}^{n} c_j = a_{oo} \\ \displaystyle\sum_{j=0}^{n} c_j\ v_j^i = a_{oi} \\ \displaystyle\sum_{j=0}^{n} c_j\ v_j^k\ v_j^\ell = a_{k\ell} \end{cases}$$

We can write (52) in the matrix form

(53) $\quad V^T\ C\ V = A$

where

$$(54)\quad \begin{cases} V = \begin{bmatrix} 1 & v_0^1 & \dots & v_0^n \\ 1 & v_1^1 & \dots & v_1^n \\ & \ddots & & \\ 1 & v_n^1 & \dots & v_n^n \end{bmatrix} \\ C = \begin{bmatrix} c_0 & 0 & \dots & 0 \\ 0 & c_1 & \dots & 0 \\ & \ddots & & \\ 0 & 0 & \dots & c_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{00} & a_{01} & \dots & a_{0n} \\ a_{01} & a_{11} & \dots & a_{1n} \\ & \dots & & \\ a_{0n} & a_{1n} & \dots & a_{nn} \end{bmatrix} \end{cases}$$

and where we assume $0 < a_{00} < \infty$ and A non-singular.

Since A is symmetric we can always find a non-singular transformation B such that

$$(55)\qquad B^T V^T C V B = B^T A B = a_{00} D$$

where D is the diagonal matrix with elements ± 1.

We set $VB = Z$, where

$$(56)\qquad Z = \begin{bmatrix} 1 & z_0^1 & \dots & z_0^n \\ 1 & z_1^1 & \dots & z_1^n \\ & \ddots & & \\ 1 & z_n^1 & \dots & z_n^n \end{bmatrix} = \begin{bmatrix} 1 & v_0^1 & \dots & v_0^n \\ 1 & v_1^1 & \dots & v_1^n \\ & \ddots & & \\ 1 & v_n^1 & \dots & v_n^n \end{bmatrix} \begin{bmatrix} 1 & b_{01} & \dots & b_{0n} \\ 0 & b_{11} & \dots & b_{1n} \\ & & & \\ 0 & b_{n1} & \dots & b_{nn} \end{bmatrix}$$

Post-multiplying (55) by $D^{-1}(VB)^T$ we obtain

$$(57)\qquad (VB)^T C (VB) D^{-1} (VB)^T = a_{00} (VB)^T \quad .$$

Pre-multiplying (57) by $C^{-1}[(VB)^T]^{-1}$ (C is non-singular) and noting that $D^{-1} = D$ we obtain

$$(58) \qquad (VB)\, D\, (VB)^T = a_{00}\, C^{-1} = Z\, D\, Z^T \quad .$$

In terms of the z^i_j this equation is

$$(59) \qquad 1 + \sum_{i=1}^{r} z^i_k\, z^i_\ell - \sum_{i=r+1}^{n} z^i_k\, z^i_\ell = \frac{a_{00}}{c_k}\, \delta_{k\ell}$$

where $r + 1$, $0 \leqq r \leqq n$ is the number of $+ 1$'s in D.

We are only interested in real solutions of (52) and therefore precisely $n - r + 1$ of the c_j must be negative by Sylvester's "law of inertia" ([9] p.56). Keeping this in mind, we first find a particular set of z^i_j's that will satisfy equations (59). We then use equation (56) to find the v^i_j's by inverting the matrix B; i.e.

$$(60) \qquad V = Z\, B^{-1} \quad .$$

From a particular solution, other solutions may be found as follows. If

$$(61) \qquad S = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & s_{11} & \cdots & s_{1n} \\ & \cdots & & \\ 0 & s_{n1} & \cdots & s_{nn} \end{bmatrix}$$

is a cogredient automorph of D, that is if $SDS^T = D$ then by equation (58) we see that

$$(62) \qquad Y = Z\, S$$

can be used instead of Z in equation (60) to obtain another solution for V.

Table I gives a particular solution of (58); we have assumed $c_0, \ldots, c_{n-r}$ negative and $c_{n+1-r}, \ldots, c_n$ positive. In cases where the double sign occurs we mean to use the upper sign for the first r components of each vector $\zeta_j = (z_j^1, \ldots, z_j^n)$ and the lower sign for the last $n - r$ components. Each z_j is real.

TABLE I

$$\zeta_0 = \left(0,\ 0,\ \ldots,\ \left[\frac{a_{00} - c_0}{\pm c_0}\right]^{\frac{1}{2}}\right)$$

$$\zeta_1 = \left(0,\ 0,\ \ldots,\ \left[\frac{a_{00}(a_{00}-c_0-c_1)}{\pm (a_{00}-c_0)c_1}\right]^{\frac{1}{2}},\ \mp\left[\frac{\pm c_0}{a_{00}-c_0}\right]^{\frac{1}{2}}\right)$$

$$\zeta_2 = \left(0,\ 0,\ \ldots,\ \left[\frac{a_{00}(a_{00}-c_0-c_1-c_2)}{\pm (a_{00}-c_0-c_1)c_2}\right]^{\frac{1}{2}},\ \mp\left[\frac{\pm a_{00}c_1}{(a_{00}-c_0)(a_{00}-c_0-c_1)}\right]^{\frac{1}{2}},\ \pm\left[\frac{\pm c_0}{a_{00}-c_0}\right]^{\frac{1}{2}}\right)$$

$$\zeta_{n-2} = \left(0,\ \left[\frac{\pm a_{00}(a_{00}-c_0-\cdots-c_{n-2})}{(a_{00}-c_0-\cdots-c_{n-3})c_{n-2}}\right]^{\frac{1}{2}},\ \ldots,\ \mp\left[\frac{\pm a_{00}c_2}{(a_{00}-c_0-c_1)(a_{00}-c_0-c_1-c_2)}\right]^{\frac{1}{2}},\right.$$

$$\left.\left[\frac{a_{00}c_1}{(a_{00}-c_0)(a_{00}-c_0-c_1)}\right]^{\frac{1}{2}},\ \mp\left[\frac{\pm c_0}{a_{00}-c_0}\right]^{\frac{1}{2}}\right)$$

$$\zeta_{n-1} = \left(\left[\frac{\pm a_{00}(a_{00}-c_0-\cdots-c_{n-1})}{(a_{00}-c_0-\cdots-c_{n-2})c_{n-1}}\right]^{\frac{1}{2}},\ \mp\left[\frac{\pm a_{00}\,c_{n-2}}{(a_{00}-c_0-\cdots-c_{n-3})(a_{00}-c_0-\cdots-c_{n-2})}\right]^{\frac{1}{2}},\right.$$

$$\left.\ldots,\ \mp\left[\frac{\pm a_{00}\,c_2}{(a_{00}-c_0-c_1)(a_{00}-c_0-c_1-c_2)}\right]^{\frac{1}{2}},\ \mp\left[\frac{\pm a_{00}\,c_1}{(a_{00}-c_0)(a_{00}-c_0-c_1)}\right]^{\frac{1}{2}},\ \mp\left[\frac{\pm c_0}{a_{00}-c_0}\right]^{\frac{1}{2}}\right)$$

$$\zeta_n = \left(\mp\left[\frac{\pm a_{00}\,c_{n-1}}{(a_{00}-c_1-\cdots-c_{n-2})c_n}\right]^{\frac{1}{2}},\ \mp\left[\frac{a_{00}\,c_{n-2}}{(a_{00}-c_0-\cdots-c_{n-3})(a_{00}-c_0-\cdots-c_{n-2})}\right]^{\frac{1}{2}},\ \ldots,\right.$$

$$\left.\mp\left[\frac{\pm a_{00}\,c_2}{(a_{00}-c_0-c_1)(a_{00}-c_0-c_1-c_2)}\right]^{\frac{1}{2}},\ \mp\left[\frac{\pm a_{00}\,c_1}{(a_{00}-c_0)(a_{00}-c_0-c_1)}\right]^{\frac{1}{2}},\ \mp\left[\frac{\pm c_0}{a_{00}-c_0}\right]^{\frac{1}{2}}\right)$$

The formulas discussed above are minimal; that is, similar formulas cannot be obtained with $m + 1$ points, where $m < n$. For if a formula could

be obtained with $m + 1$ points X_j, $j=0,\ldots,m$, $m < n$ then equation (53) would still hold where A is the same as before and U is $(m+1)x(n+1)$, C is $(m+1)x(m+1)$. Hence $\det (U^T CU) = 0$ since $\det U = 0$. By assumption $\det A \neq 0$ and thus (53) cannot hold if $m < n$.

Proceeding in this manner, A. H. Stroud also obtained a 2n-point precision 3 integration formula valid over a symmetric but otherwise arbitrary region in E^n. In this case the weight function is also symmetric, but otherwise arbitrary. In the same paper he gave results for the precision 3 case similar to those in Table I and he also gave methods for extending these results to other regions in E^n by linear transformations.

ix) <u>Monte Carlo method</u>

A very well known method of numerical integration applicable over an arbitrary region in E^n is the Monte Carlo method. This method essentially consists of the evaluation of

$$(63) \qquad \int_{R_n} f(X)\, dX = \frac{V}{m} \sum_{j=1}^{m} f(X_j) \; ; \quad V = \int_{R_n} dX$$

where the X_j's are points chosen uniformly at "random" throughout the region of integration, R_n.

If $f(X)$ is sufficiently well behaved (e.g. of bounded variation in the variables X) it can be shown statistically that the error of integration

$$(64) \qquad \left| \frac{V}{m} \sum_{j=1}^{m} f(X_j) - \int_{R_n} f(X)\, dX \right| = O\left(\frac{1}{\sqrt{m}}\right) \quad .$$

We have quotations around the word <u>random</u> above since the points X_j are obtained by the use of a formula, for example

$$x_{i+1} = (\alpha x_i + \beta) \bmod k.$$

As an example of the accuracy obtainable by this method, Davis and Rabinowitz (Vol. 10, p.1, M.T.A.C.) show they can obtain results to within one per cent error using approximately 10,000 points to compute the volume of a 4-dimensional sphere. Other examples are given on page 48 of this thesis.

Suppose that due to a large value of n and complexity of $f(X)$ it takes an I.B.M. 7090 one hour to compute a solution accurate to one significant figure. To achieve 4-figure accuracy would then take over 20 years.

Due to convergence to an accurate solution being slow by the Monte Carlo method, the method is used normally to make rough estimates of integrals of a dimension so high that it would be impractical using a numerical integration formula.

For a detailed explanation of Monte Carlo methods see the paper "Monte Carlo methods for solving multivariable problems", by J. M. Hammersley in Annals of the N.Y. Acad. Sc. 1960 Vol. 86, Art. 3, p. 845-874. This paper also contains a wealth of references on the method.

x) Taylor Series approach

J.C.P. Miller has written several papers ([10],[11],[12]) on integration over a rectangular domain. To obtain an integration formula of the form

$$(65) \qquad I = \int_{-h}^{h} \cdots \int_{-h}^{h} f(X)\, dX \cong (2h)^n \sum_{j=1}^{m} c_j\, f(X_j)$$

he expands an arbitrary function $f(X)$ in a Taylor series in n dimensions and tries to chose his m points over the region of integration (the n-cube) in such a way as to represent the function by as many of the first terms of

the Taylor series expansion as possible.

Integrating his Taylor series expansion over the n-cube of side h he obtains

$$(66) \qquad I = (2h)^n \left\{ f_0 + \frac{h^2}{3!} S_2 + \frac{h^4}{5!}(S_4 + S_{2,2}) + \frac{h^6}{7!}(S_6 + S_{4,2} + S_{2,2,2}) + \frac{h^8}{9!}(S_8 + S_{6,2} + S_{4,4} + S_{4,2,2} + S_{2,2,2,2}) + \cdots \right\}$$

where by f_0 we mean $f(0,\ldots,0)$ and where, for example

$$(67) \qquad S_{2,2,2} = s_{2,2,2} \sum_{i,j,k=1}^{n}{}^{*} \frac{\partial^6 f_0}{(\partial x^i)^2(\partial x^j)^2(\partial x^k)^2} ,$$

the asterisk indicating that i, j, k are unequal in pairs throughout, $a_{2,2,2} = 7!/3!\ 3!\ 3!$.

Setting up and solving non-linear algebraic equations he obtains various integration formulas; some of these being similar to those that have previously been obtained by Hammer and Stroud, and which we will list later. His procedure also gives us an error estimate depending on high order partial derivatives of the function $f(X)$, and although such an error estimate may be difficult to evaluate, his papers are among a very few that discuss errors of numerical integration formulas in higher dimensions at all.

Moreover, by his method he is able to see that harmonic functions require fewer points of evaluation than arbitrary functions, and he exploits this in [12] using polynomials orthogonal over the interval $(0,1)$ with respect to weight functions $\frac{1}{2}(x^{-3/2} - x^{-1/2})$, $\frac{1}{2}(x^{1/4} - x^{1/2})$. As an example he evaluates

$$\int_{-1}^{1}\int_{-1}^{1} \cos x \cosh y \, dx\, dy$$

to an accuracy of ten significant figures using only five points on the square. (We recall here, that both $\cos x$ and $\cosh x$ have very rapidly converging Taylor series expansions, and hence this is probably an example of the integration formula at its best. The accuracy of the result is surprising, nevertheless).

xi) <u>The error bound of von Mises</u>

We close this chapter with a summary of an error bound due to von Mises [26], and summarized by Stroud [27].

Suppose we have the integration formula

(68) $$\int_{R_n} f(X)\, dX = \sum_{j=1}^{m} c_j\, f(X_j) + E$$

where R_n is starlike* with respect to an interior point Z. Transforming to spherical coordinates, von Mises shows that

(69)
$$\left\{\begin{array}{l} |E| \leq |f^{(\mu)}|_{\max} \dfrac{1}{\mu!} \left[\displaystyle\int_{R_n} r^{\mu}\, dX + \sum_{j=1}^{m} |c_j|\, r_j^{\mu}\right] \\ \text{where} \\ r = [(x^1-z^1)^2 + \dots + (x^n-z^n)^2]^{\frac{1}{2}} \\ f^{(\mu)} = \left[\dfrac{x^1-z^1}{r}\dfrac{\partial}{\partial x^1} + \dots + \dfrac{x^n-z^n}{r}\dfrac{\partial}{\partial z^n}\right]^{\mu} f(X)\ . \end{array}\right.$$

Here if the integration formula has precision p then μ can be taken to be $1,2,\dots,p+1$; r is the distance from Z, and r_j is the distance of the j'th point from Z.

---

* A region R_n in E^n is starlike with respect to an interior point Z if any half ray emanating from Z cuts the boundary of R_n at one and only one point.

For even values of $\mu \leqq p$ the integral $\int_{R_n} r^{\mu}\, dX$ will be known from the monomial integrals required to determine the integration formula. For odd values of μ this integral may be difficult to evaluate, and in such cases it may be desirable to use the Schwarz inequality

$$(70) \qquad \int_{R_n} r^{\mu}\, dX \leqq \left(V(R_n) \int_{R_n} r^{2\mu}\, dX \right)^{\frac{1}{2}}$$

where $V(R_n)$ is the volume of R_n .

If the c_j are all positive, then von Mises shows that equation (69) can be improved to

$$(71) \qquad \begin{aligned} |E| &\leqq \frac{1}{\mu!} \left[f^{(\mu)}_{max} \int_{R_n} r^{\mu}\, dX - f^{(\mu)}_{min} \sum_{j=1}^{m} c_j\, r_j^{\mu} \right] \\ |E| &\geqq \frac{1}{\mu!} \left[f^{(\mu)}_{min} \int_{R_n} r^{\mu}\, dX - f^{(\mu)}_{max} \sum_{j=1}^{m} c_j\, r_j^{\mu} \right] . \end{aligned}$$

Equations (69) and (71) are valuable theoretical results, but finding the derivative of a complicated integrand and obtaining a bound on it can be extremely difficult in practical cases, particularly for large n and/or p.

xii) Summary

The theory and results presented up to this point constitute a rough outline of current methods of attack.

To restate the problem, we should like to have numerical integration formulas of the type (4) of Chapter I which employ values of the integrand at points lying within the region of integration. The formulas are to have precision p as described on page 3 and the total number of points m is to be as small as possible.

The solution of this problem faces us with another problem; the solution of a set of simultaneous non-linear algebraic equations*. Hence if effective methods of solving a set of simultaneous non-linear algebraic equations were available, the task of finding numerical integration formulas in higher dimensions would not be such a difficult problem.

In setting up these non-linear algebraic equations it would help considerably if we knew in advance the minimum number of points $m = m(n,p,R_n,w)$ required to obtain integration formulas of the type (4) of Chapter I. We already know from a study of integration formulas in one dimension that this minimum number of points may be dependent on $w(X)$. There is some evidence that this minimum number of points may also be region dependent. To illustrate this point we have presented above the derivation of an $n + 2$ - point precision 3 integration formula for the n-simplex (Hammer and Stroud [6]), while the best existing formula of precision 3 uses $2n$ points (it is not known whether this is a minimal formula). Thacher [22] claims, moreover, that a 5-point precision 3 formula for the cube does not exist.

It is desirable to keep n arbitrary, since we wish to achieve general, rather than particular integration formulas in higher dimensions. Although we lose somewhat in generality** if we restrict our considerations to fully symmetric regions we gain in algebraic convenience since we shall have fewer non-linear equations to solve and the number of these equations will not depend on n.

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* Workers in research are faced with the non-linear problem far more frequently than they are faced with the linear one.

** The integration formulas may undergo a loss of precision under transformation (transformation theorem, Hammer et alii p.9).

Error estimates are necessary. In one dimension, the error of integration formulas is popularly expressed in the form of a high order derivative of the integrand--obtaining a bound on an error estimate such as this may be extremely difficult, and this is much more so in higher dimensions.

CHAPTER III

THE CONSTRUCTION OF INTEGRATION FORMULAS

In this chapter we compare the usefulness of formulas which integrate all monomials of the form $\prod_{i=1}^{n} (x^i)^{k_i}$, $\sum_{i=1}^{n} k_i \leq p$ with formulas which integrate all monomials of the form $\prod_{i=1}^{n} (x_i)^{k_i}$, $k_i \leq p$ (the formulas that are Cartesian products of formulas in one variable). The minimum number of points required to obtain an integration formula of precision p over an arbitrary region in n-space is estimated.

Starting with appropriate sets of symmetrically spaced points non-linear algebraic equations are set up and solved to obtain fully symmetric numerical integration formulas.

Orthogonal polynomials are constructed, and as for the case of one-dimensional integration the zeros of these polynomials (at least the zeros of the odd polynomials!) turn out to be the required non-linear unknowns in the simultaneous non-linear algebraic equations set up to obtain numerical integration formulas. The polynomials were thus an aid in obtaining the solution to the non-linear algebraic equations.

1) <u>A Discussion of Precision</u>

We recall (see Cartesian product theorem Chapter II, p. 11) that high precision integration formulas are available over Cartesian product regions in n-space. For example, to obtain a formula of precision $2m-1$ we would need m points on a line, m^2 points on a rectangle, ..., m^n points on a rectangular region in n-space. To obtain a formula of precision 3 over the n-cube we would need 2^n points, while formula (10) of Chapter I tells us that we can obtain precision 3 with only $2n$ points.

This is a somewhat unfair comparison however, since a repeated Gaussian integration formula will integrate so many more monomials than an integration formula which will only integrate monomials up to degree p. A repeated Gaussian integration formula will integrate any monomial of the form

$$(1) \qquad \prod_{i=1}^{n} (x^i)^{k_i} \quad \text{where} \quad 0 \leqq k_i \leqq p \ .$$

This monomial is of degree $\leqq p$ in each variable and of total degree $\leqq n\,p$.

The definition of precision (an integration formula is said to have precision p if it exactly integrates any monomial of the form $\prod_{i=1}^{n} (x^i)^{k_i}$ where $0 \leqq \sum_{i=1}^{n} k_i \leqq p$), then has made a whole series of integration formulas possible; formulas which at first notice seem to perform as well as the repeated Gaussian formulas, but require by far fewer points. Now it is true that the formulas arising as a result of this definition of precision require fewer points than the repeated Gaussian integration formulas, but this is mainly because the former formulas integrate fewer monomials than the latter. With a precision 3, 2^n - point repeated Gaussian integration formula we can exactly integrate 4^n different monomials, while with the $2n$ - point formula (10) of Chapter I we can exactly integrate $(n+1)(n+2)(n+3)/3!$ different monomials. For $n = 20$, the ratio of the number of monomials exactly integrated by the two formulas is roughly 10^{12} to $2x\ 10^3$.

It is also interesting to compare the number of monomials each of these formulas integrate per point. Any repeated Gaussian integration formula will integrate 2^n monomials per point, while formula (10) of Chapter I will integrate only $0(n^2)$ monomials per point. We conjecture that 2^n monomials per point is the best possible.

The polynomial

$$(2) \qquad \prod_{i=1}^{n} \left[\sum_{j=0}^{k_i} a_{i_j} (x^i)^j \right] \ , \quad 0 \leqq k_i \leqq p$$

has monomials up to degree p in each independent variable. That is, it contains all the monomials an arbitrary "Cartesian product" type polynomial

of degree p would contain. The polynomial

$$(3) \qquad \prod_{j=1}^{p} \left[a_{0_j} + \sum_{i=1}^{n} a_{i_j} x^i \right]$$

contains all the monomials an arbitrary Taylor polynomial of degree p would have.

Now any linear or affine transformation will not alter the degree of (3). However, a linear or affine transformation will increase the degree of (2), unless the transformation is of the form $\bar{x}^i = a^i x^i + b^i$. This seems to indicate that the integration points for a repeated Gaussian integration formula are uniquely determined. It suggests, moreover (as has already been shown for a particular case on page 23) that the points x^i_j of integration formulas capable of integrating only the monomials up to degree p (and not products of monomials up to degree p in each independent variable) are not uniquely determined.

When integrating a monomial of the type (5) of Chapter I, we integrate with respect to each variable x^i holding the remaining variables fixed. Hence in this application we are concerned more with the highest degree in each variable than with the total degree in all the variables, i.e. the degree k_i in each x^i is more significant than the total degree $\sum_{i=1}^{n} k_i$.

All the above arguments, together with the difficulty of obtaining interpolation polynomials that represent an arbitrary polynomial of degree p in n variables (see H. C. Thacher Jr., New York Acad. Sci., v.86, art.3, p.758-775) seem to indicate that the definition of precision given on page 3 of this thesis does not fit too well, particularly for integration formulas over rectangular regions in hyperspace.

We must, nevertheless, not forget that the total number of points required in a repeated Gaussian integration formula becomes fantastically large as n increases. It may, moreover, often occur that we can obtain comparable accuracy using a formula that requires by far fewer points. To illustrate this, let us estimate the number of points m required to obtain an integration formula of precision p over an arbitrary region in n-space.

We have at our disposal a choice of m weights and mn coordinates to fit an arbitrary polynomial of degree p in n-space that has $\binom{n+p}{p}$ monomials. Hence $m(n+1) \geqq \binom{n+p}{p}$, or

$$(4) \qquad m \geqq \frac{(n+p)!}{p!\,(n+1)!}$$

When n is large we may use Stirling's formula

$$(5) \qquad \Gamma(n+p+1) \sim \sqrt{2\pi}\, n^{n+p+\frac{1}{2}}\, e^{-n} \qquad \text{to obtain}$$

$$(6) \qquad m \geqq \frac{n^{p-1}}{p!}$$

as an estimate of the minimum number of points m required to obtain an integration formula of degree p over a region in n-space*. The repeated Gaussian formulas require $(\frac{p+1}{2})^n$ points.

In the Table on the following page we have tabulated the minimum number of points required by a formula of precision p along with the number of points required by a Gaussian formula of precision p.

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* For an improvement of this bound to $m \geqq n^k/k!$, k an integer, $p = 2k$ see [27]; the fully symmetric formulas of the following sections require $m = \frac{k\, 2^{2k}\, n^{2k}}{(2k)!}\,[1 + O(\frac{1}{n})]$ points for large n and precisions $p = 4k+1$.

TABLE II

n	p	Minimum required $\frac{(n+p)!}{p!(n+1)!}$	Gaussian $(\frac{p+1}{2})^n$
4	5	26	81
4	9	143	625
4	11	273	1,296
4	19	1,771	10,000
8	9	2,601	390,625
8	19	247,000	100,000,000

This Table clearly indicates that the Gaussian integration formulas reach the economically feasible point much sooner than formulas that use a minimum number of points for even the fastest computers. We would therefore like to suggest that if we had "minimum point" formulas we could in most cases obtain accurate results with these at a lesser cost than with repeated Gaussian formulas.

2) The Setting up and Solution of Non-linear Algebraic Equations

The approach given here is similar to that given in [7] by Hammer and Stroud. In looking for fully symmetric integration formulas* we first chose appropriate sets of symmetrically spaced points. The coordinate sets and the number of points obtainable with each are tabulated below. The method of counting these is as follows.

Suppose we have k different coordinates: r_1 of the first, ..., r_k of the kth so that $\sum_{j=1}^{k} r_j = n$. Then by rearranging these coordinates in all possible ways we can obtain $n! / \prod_{j=1}^{k} r_j!$ different points in n-space. If we further allow all possible changes of signs of the coordinates we can obtain $2^q n! / \prod_{j=1}^{k} r_j!$ different points in n-space, where we have assumed

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* For the definition of a fully symmetric integration formula see page 8.

$q \leqq n$ non-zero coordinates.

Coordinates	Restriction on n	Number of Points
$(0,\ldots,0)$	$n \geqq 1$	$2^0\binom{n}{0}$
$(\pm u,0,\ldots,0)$	$n \geqq 1$	$2^1\binom{n}{1}$
$(\pm u,\pm u,0,\ldots,0)$	$n \geqq 2$	$2^2\binom{n}{2}$
$(\pm u,\pm v,0,\ldots,0)$	$n \geqq 2$	$2^2\binom{n}{2}2!$
$(\pm u,\pm u,\pm u,0,\ldots,0)$	$n \geqq 3$	$2^3\binom{n}{3}$
$(\pm u,\pm u,\pm u,\pm u,0,\ldots,0)$	$n \geqq 4$	$2^4\binom{n}{4}$

Stroud in v.14, p.21, M.T.A.C. gives precision 3 integration formulas valid over an arbitrary symmetric region in E^n. Note the distinction between symmetric and fully symmetric. A cylinder for example is symmetric, while the n-cube is both symmetric and fully symmetric.

The integration formulas we will try to obtain are for fully symmetric regions. The weight function of these integration formulas is also assumed to be fully symmetric, but otherwise arbitrary.

From the definition of fully symmetric it follows that:

(a) the integral over a fully symmetric region R of any product of the coordinate variables which contains an odd power is zero; and

(b) the integral of a product of even powers depends only on their exponent and not on their ordering.

In what follows we have written down non-linear algebraic equations and their solutions to obtain fully symmetric integration formulas of precision 3, 5 and 9. The method of solving the non-linear algebraic equations is given in the next section. On the right hand side of the non-linear algebraic equations we designate $\int_{R_n} w(X)(x^i)^2(x^j)^4\,dX$, $i \neq j$ for example by

$I_{24}(= I_{42})$. We assume also that R_n contains the origin*.

i) Formulas of precision 3

To obtain the 2n-point equi-weighted integration formula of precision 3,

$$I(f) \equiv \int_{R_n} w(X)\, f(X)\, dX \cong A \sum_{j=1}^{2n} f(\pm u,0,\ldots,0) \;,$$

we solve the following pair of simultaneous non-linear algebraic equations:

$$(7) \qquad \begin{cases} 2n\,A & I_0 \\ 2A\,u^2 = I_2 \end{cases} \quad .$$

The first equation here is obtained by substituting the integral of a constant into the above integration formula; the second by substituting the integral of the square of a particular variable.

General Solution. We easily obtain the value of A from the first of the above non-linear equations; the second equation then gives us the value of u. The general solution is

$$A = I_0/2n \;, \quad u = (nI_2 / I_0)^{1/2} \quad .$$

Four particular solutions are tabulated in Table III.

ii) Formulas of precision 5

We assume that we can obtain a $2n^2 + 1$ - point formula of the form

$$I(f) \cong A\, f(0,\ldots,0) + B \sum_{1}^{2n} f(\pm u,0,\ldots,0) + C \sum_{1}^{2n(n-1)} f(\pm u,\pm u,0,\ldots,0)$$

Due to our choice of a symmetric set of points all odd powers of monomials

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* It is felt that this assumption may not be necessary.

will sum to zero. To obtain the values for u, A, B, C we need to solve the following system of non-linear algebraic equations

$$(8)\quad \left\{ \begin{bmatrix} 1 & 2n & 2n(n-1) \\ 0 & 2u^2 & 4(n-1)u^2 \\ 0 & 2u^4 & 4(n-1)u^4 \\ 0 & 0 & 4u^4 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \\ 0 \end{bmatrix} = \begin{bmatrix} I_0 \\ I_2 \\ I_4 \\ I_{22} \end{bmatrix} \right. .$$

<u>General Solution</u>. The value of u is easily obtained by dividing the second equation into the third. By ordinary elimination we can then obtain the value of the linear coefficients. The general solution is

$$A = I_o - n(I_2/I_4)^2 \left[I_4 - \frac{n-1}{2} I_{22}\right]$$

$$B = \frac{1}{2} (I_2/I_4)^2 \left[I_4 - (n-1) I_{22}\right]$$

$$C = \frac{1}{4} (I_2/I_4)^2$$

$$u = (I_4/I_2)^{1/2} \quad .$$

Note that we have assumed $n \geqq 2$. Four particular solutions are tabulated along with the precision 3 formulas in Table III.

iii) <u>Formulas of precision 9</u>

Due to difficulties involved which will be made clear in the following sections we have not obtained the formulas of precision 7. We have, however, obtained the formulas of precision 9. In the presentation of these formulas we use the notations

$$n^{(k)} = n(n-1)\ldots(n-k+1)$$

$$n_{(k)} = \frac{n(n-1)\ldots(n-k+1)}{k!} \quad .$$

TABLE III

Precision	Zeros and Weights	n - Cube $w(X) = 1$	n - Sphere $w(X) = 1$	n - Cube $w(X) = 1/\prod_{i=1}^{n} [1-(x^i)^2]^{1/2}$	Infinite n-space $w(X) = \exp\left[-\sum_{i=1}^{n} (x^i)^2\right]$
3	u	$(n/3)^{1/2}$	$(\frac{n}{n+2})^{1/2}$	$(n/2)^{1/2}$	$(n/2)^{1/2}$
	A	$I_0/2n$	$I_0/2n$	$I_0/2n$	$I_0/2n$
5	u	$(3/5)^{1/2}$	$[3/(n+4)]^{1/2}$	$(3/4)^{1/2}$	$(3/2)^{1/2}$
	A	$\frac{I_0}{162}(25n^2-115n+162)$	$I_0 \frac{(n^3-3n^2-10n+36)}{18n+36}$	$\frac{I_0}{9}(2n^2 - 8n + 9)$	$\frac{I_0}{18}(n^2 - 7n + 18)$
	B	$\frac{I_0}{162}(70 - 25n)$	$I_0 \frac{16 - n^2}{18n + 36}$	$\frac{I_0}{9}(5 - 2n)$	$\frac{I_0}{18}(4 - n)$
	C	$\frac{I_0}{324}(25)$	$I_0 \frac{n + 4}{36n + 72}$	$I_0/9$	$I_0/36$
I_0		2^n	$\frac{\pi^{n/2}}{\Gamma(1+n/2)}$	π^n	$\pi^{n/2}$

The formulas are assumed to have the form

$$I(f) \cong A\, f(0,\dots,0) + \sum_{1}^{2n} [B\, f(\pm u,0,\dots,0) + C\, f(\pm v,0,\dots,0)]$$

$$+ \sum_{1}^{2^2 n^{(2)}} [D\, f(\pm u,\pm u,0,\dots,0) + E\, f(\pm v,\pm v,0,\dots,0)]$$

$$+ \sum_{1}^{2^2 n^{(2)}} F\, f(\pm u,\pm v,0,\dots,0)$$

$$+ \sum_{1}^{2^3 n^{(3)}} [G\, f(\pm u,\pm u,\pm u,0,\dots,0) + H\, f(\pm v,\pm v,\pm v,0,\dots,0)]$$

$$+ \sum_{1}^{2^4 n^{(4)}} [I\, f(\pm u,\pm u,\pm u,\pm u,0,\dots,0) + J\, f(\pm v,\pm v,\pm v,\pm v,0,\dots,0)] .$$

To obtain the values of u, v, A, ..., J of this $\frac{4}{3}(n^4 - 4n^3 + 11n^2 - 7n) + 1$ - point* formula we need to solve the following system of non-linear algebraic equations:

(9) See page 45.

General Solution. This is valid provided

(10) $$I_{62} + u^2v^2 I_{22} = (u^2 + v^2)\, I_{42} \;** .$$

The method by which this solution was obtained is given later.

(11) $$u,v = \left[\frac{I_2I_8 - I_4I_6 \pm [(I_2I_8)^2 + 4I_4{}^3 I_8 + 4I_2I_6{}^3 - 6I_2I_4I_6I_8 - 3(I_4I_6)^2]^{1/2}}{2(I_2I_6 - I_4I_4)}\right]^{1/2}$$

$$F = \frac{I_{62} - I_{44}}{4u^2v^2(u^2 - v^2)^2}$$

*G = 0 This reduces the number of integration points to $\frac{4}{3}(n^4 - 5n^3 + 14n^2 - 7n) + 1$.

** So far we have found no examples for which this equation does not hold.

$$
(9)\quad
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & \frac{u^2}{n} & \frac{v^2}{n} & \frac{2u^2}{n} & \frac{2v^2}{n} & \frac{2u^2}{n} & \frac{2v^2}{n} & \frac{3u^2}{n} & \frac{3v^2}{n} & \frac{4u^2}{n} & \frac{4v^2}{n} \\
0 & \frac{u^4}{n} & \frac{v^4}{n} & \frac{2u^4}{n} & \frac{2v^4}{n} & \frac{2u^4}{n} & \frac{2v^4}{n} & \frac{3u^4}{n} & \frac{3v^4}{n} & \frac{4u^4}{n} & \frac{4v^4}{n} \\
0 & \frac{u^6}{n} & \frac{v^6}{n} & \frac{2u^6}{n} & \frac{2v^6}{n} & \frac{2u^6}{n} & \frac{2v^6}{n} & \frac{3u^6}{n} & \frac{3v^6}{n} & \frac{4u^6}{n} & \frac{4v^6}{n} \\
0 & \frac{u^8}{n} & \frac{v^8}{n} & \frac{2u^8}{n} & \frac{2v^8}{n} & \frac{2u^8}{n} & \frac{2v^8}{n} & \frac{3u^8}{n} & \frac{3v^8}{n} & \frac{4u^8}{n} & \frac{4v^8}{n} \\
0 & 0 & 0 & \frac{2u^4}{n_{(2)}} & \frac{2v^4}{n_{(2)}} & \frac{2u^2v^2}{n_{(2)}} & \frac{2u^2v^2}{n_{(2)}} & \frac{6u^4}{n_{(2)}} & \frac{6v^4}{n_{(2)}} & \frac{12u^4}{n_{(2)}} & \frac{12v^4}{n_{(2)}} \\
0 & 0 & 0 & \frac{2u^6}{n_{(2)}} & \frac{2v^6}{n_{(2)}} & \frac{2u^4v^2}{n_{(2)}} & \frac{2u^2v^4}{n_{(2)}} & \frac{6u^6}{n_{(2)}} & \frac{6v^6}{n_{(2)}} & \frac{12u^6}{n_{(2)}} & \frac{12v^6}{n_{(2)}} \\
0 & 0 & 0 & \frac{2u^8}{n_{(2)}} & \frac{2v^8}{n_{(2)}} & \frac{2u^4v^4}{n_{(2)}} & \frac{2u^4v^4}{n_{(2)}} & \frac{6u^8}{n_{(2)}} & \frac{6v^8}{n_{(2)}} & \frac{12u^8}{n_{(2)}} & \frac{12v^8}{n_{(2)}} \\
0 & 0 & 0 & \frac{2u^8}{n_{(2)}} & \frac{2v^8}{n_{(2)}} & \frac{2u^6v^2}{n_{(2)}} & \frac{2u^2v^6}{n_{(2)}} & \frac{6u^8}{n_{(2)}} & \frac{6v^8}{n_{(2)}} & \frac{12u^8}{n_{(2)}} & \frac{12v^8}{n_{(2)}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{6u^6}{n_{(3)}} & \frac{6v^6}{n_{(3)}} & \frac{24u^8}{n_{(3)}} & \frac{24v^8}{n_{(3)}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{6u^8}{n_{(3)}} & \frac{6v^8}{n_{(3)}} & \frac{24u^8}{n_{(3)}} & \frac{24v^8}{n_{(3)}} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{24u^8}{n_{(4)}} & \frac{24v^8}{n_{(4)}}
\end{bmatrix}
\begin{bmatrix}
A \\ n_{(1)}B \\ n_{(1)}C \\ n_{(2)}D \\ n_{(2)}E \\ n_{(2)}F \\ n_{(2)}F \\ n_{(3)}G \\ n_{(3)}H \\ n_{(4)}I \\ n_{(4)}J \\ 0
\end{bmatrix}
=
\begin{bmatrix}
I_0 \\ I_2 \\ I_4 \\ I_6 \\ I_8 \\ I_{22} \\ I_{42} \\ I_{44} \\ I_{62} \\ I_{222} \\ I_{422} \\ I_{2222}
\end{bmatrix}
\quad
\begin{array}{l}
\text{(i)} \\ \text{(ii)} \\ \text{(iii)} \\ \text{(iv)} \\ \text{(v)} \\ \text{(vi)} \\ \text{(vii)} \\ \text{(viii)} \\ \text{(ix)} \\ \text{(x)} \\ \text{(xi)} \\ \text{(xii)}
\end{array}
$$

$$H = \frac{I_{422} - (n-3)I_{2222}}{8v^8}$$

$$I = \frac{I_{422} - v^2 I_{222}}{16(n-3)u^6(u^2-v^2)}$$

$$J = \frac{I_{2222} - 16Iu^8}{16v^8}$$

$$E = \frac{u^2 I_{22} - I_{42}}{4v^4(u^2-v^2)} - Fu^2/v^2 - 2(n-2)\ [H + (n-3)J]$$

$$D = \frac{I_{42} - v^2 I_{22}}{4u^4(u^2-v^2)} - Fv^2/u^2 - 2(n-2)(n-3)I$$

$$C = \frac{u^2 I_2 - I_4}{2v^2(u^2-v^2)} - 2(n-1)\{E + F + (n-2)\ [H + \tfrac{2}{3}(n-3)J]\}$$

$$B = \frac{I_4 - v^2 I_2}{2u^2(u^2-v^2)} - 2(n-1)\ [D + F + \tfrac{2}{3}(n-2)(n-3)I]$$

$$A = I_0 - 2n(B + C + (n-1)\{D + E + 2F + \tfrac{1}{3}(n-2)[2H + (n-3)(I+J)]\})$$

For the n-cube, $w(X) = 1$: $u,v = \{5/9[1 \pm (8/35)^{1/2}]\}^{1/2}$. For the n-sphere, $w(X) = 1$: $u,v = \{5/(n+8)[1 \pm [(2n+6)/(5n+30)]^{1/2}]\}^{1/2}$. For the n-cube, $x(X) = \prod_{i=1}^{n} (1/\sqrt{1 - (x^i)^2})$: $u,v = \{5/8[1 \pm (1/5)^{1/2}]\}^{1/2}$. For infinite n-space, $w(X) = \exp(-\sum_{i=1}^{n} (x^i)^2)$: $u,v = \{5/2[1 \pm (2/5)^{1/2}]\}^{1/2}$.

Notice that we can obtain <u>two</u> precision nine integration formulas by interchanging the role of u and v in the solution to the above non-linear algebraic equations.*

---

* Actually we could have obtained an infinite number of solutions by putting $G = kH$, k an arbitrary constant. In this case we could not take advantage of the reduction of points occuring as a result of putting $G = 0$.

The numerical values of the coefficients and zeros of the precision three and five integration formulas are readily evaluated by computer. In Table V (Appendix) we list zeros and weights of particular precision nine integration formulas we previously discussed. In Appendix A we give examples to illustrate the accuracy of integration formulas developed in this thesis.

Table IV is an excerpt from Stroud [21] to which we have added our own results. This Table shows the results of some approximate calculations for the two integrals

$$\text{(I)} \qquad \left(\int_0^1\right)^4 e^{x^1x^2x^3x^4}\, dx^1dx^2dx^3dx^4$$

$$\text{(II)} \qquad \left(\int_0^1\right)^4 \left(|x^1-\tfrac{1}{2}| + |x^2-\tfrac{1}{2}| + \|x^3-\tfrac{1}{2}| + |x^4-\tfrac{1}{2}|\right) dx^1dx^2dx^3dx^4 \quad .$$

These integrals were calculated numerically by the following methods:

METHOD A. The integral is approximated by

$$\frac{1}{m}\sum_{j=1}^{m} f(X_j)$$

where** $X_j = ([j\sqrt{2}/2],\ [j\sqrt{3}],\ [j\sqrt{6}/3],\ [j\sqrt{10}])$.

The calculations for integral I were given by Davis and Rabinowitz (see page 29). The calculations for II were given by Stroud [21].

METHOD B. The formula of precision 2 given by Stroud (see page 19).*

METHOD C. The formula of precision 3 given by Stroud (see page 10).

METHOD D. The formula of precision 5 given by Hammer and Stroud [7].

METHOD E. The Cartesian product of the two point formula for the line segment.

METHOD F and G. The formulas of precision 9 derived in this thesis.

** The symbol [Y] for example indicates the fractional part of the number Y.

* These formulas are also given in this thesis.

TABLE IV

COMPARISON OF A QUASI-MONTE CARLO METHOD WITH FIVE QUADRATURE FORMULAS

METHOD	m	I (VALUES OF INTEGRALS)	II
	Exact value	1.0693976	1.0000000
A	4	1.0646192	.8523646
	8	1.0592766	.9654735
	16	1.0615567	.9597948
	32	1.0626119	.9837211
	64	1.0586261	.9915821
	128	1.0657314	.9936179
	256	1.0673119	.9977914
	512	1.0668403	.9998982
	1024	1.0681499	-------
	2048	1.0685418	-------
	4096	1.0685545	-------
	8192	1.0688021	-------
	16384	1.0691568	-------
	32768	1.0691964	-------
B	5	1.0686301	1.0310313
C	8	1.0622464	.9855997
D	33	1.0688327	.8606629
E	16	1.0693883	1.1547005
F	177	1.0694046	.9448504
G	177	1.0693986	.9448505

The integrand in II has a discontinuous derivative in the range of integration and is therefore weighted heavily against the formulas derived in this thesis and the Gaussian formulas. Hence, although the 8-point formula C gives better results than the 33-point formula and the 177-point formulas this will not generally be so.

3) Orthogonal Polynomials

By our method of setting up non-linear algebraic equations we obtain numerical integration formulas of precision p over fully symmetric regions in E^n, where $n \geqq (p-1)/2$. We shall prove the validity of the solution to

the set of equations (9) by presenting another approach to the problem of obtaining numerical integration formulas over fully symmetric regions in E^n; this approach will prove to be an aid in finding numerical integration formulas of precision $4k + 1$, k an integer.

i) <u>Polynomials orthogonal over the n-cube; $w(X) = 1$</u>.

As an aid in finding integration formulas that have some maximum precision, we construct a set of polynomials orthogonal over the n-cube according to the following rules:

(a) All the independent variables are considered equally important. For example, in the case $n = 2$, if the orthogonal polynomial has the term kx^2, it shall also have the term ky^2 (i.e. the polynomials are symmetric in the variables).

(b) The integral of the product of a monomial degree less than m and an orthogonal polynomial of degree m over the region R_n shall be zero.

By these rules we are merely trying to extend the idea of orthogonal polynomials in one variable.

In trying to construct a set of orthogonal polynomials in higher dimensions according to these two rules we find that there exists no unique set of polynomials. For example each of the bases $1, x, y, x^2, y^2, \ldots, x^k, y^k, \ldots,$ or $1, xy, \ldots, (xy)^k, \ldots$ could be used to construct a set of polynomials orthogonal over the square. Once we have constructed a particular set of orthogonal polynomials we may moreover find that the zeros of each member of the set are not uniquely determined, since they are level curves. The polynomial $x + y$, for example has an infinite number of zeros, $x = -y$.

We will call this set of polynomials $C_m^n(X)$; C for cube, m being the degree and n the dimension. To preserve total degree the pair

$C_0^n(X) = 1, \quad C_1^n(X) = \sum_{i=1}^{n} x^i$ is an obvious one to start with. From these two it is easy to see that all other orthogonal members of the set will have monomials with either odd degree or even degree, but not both.

It is not immediately obvious which one of $C_2^n(X) = 1 + a \sum_{i=1}^{n}\sum_{j=1}^{i} x^i x^j$, $C_2^n(X) = 1 + a\sum_{i=1}^{n} (x^i)^2 + b\sum_{j\neq 1} x^i x^j$, or $1 + a\sum_{i=1}^{n} (x^i)^2$ we should take. We cannot solve for b in the second case. Consideration of $C_3^n(X)$ tells us nothing we don't already know, so let us consider $C_4^n(X)$ for $n = 2$. Here we have the following choices:

$$\text{(i)} \quad C_4^2(X) = 1+a(x^2+y^2+xy)+b(x^4+x^3y+x^2y^2+xy^3+y^4)$$

$$\text{(ii)} \quad C_4^2(X) = 1+a(x^2+y^2)+cxy+b(x^4+y^4)+ex^2y^2+d(x^3y+xy^3)$$

$$\text{(iii)} \quad C_4^2(X) = 1+a(x^2+y^2)+b(x^4+y^4)+ex^2y^2$$

$$\text{(iv)} \quad C_4^2(X) = 1+a(x^2+y^2)+b(x^4+y^4) \quad .$$

All the constants can be uniquely determined in only the last of these choices. Hence if we further add that the polynomials should be the simplest possible that satisfy (a) and (b) above, we find these polynomials to be

$$(10) \quad \begin{cases} C_{2m}^n(X) = 1 + \sum_{i=1}^{n}\sum_{j=1}^{m} a_j (x^i)^{2j} \\ C_{2m+1}^n(X) = \sum_{i=1}^{n} (x^i + \sum_{j=1}^{m} a_j (x^i)^{2j+1}) \quad . \end{cases}$$

By simplest possible we mean that the polynomials have the fewest possible monomials required to satisfy rules (a) and (b) above.

By evaluating

$$(11) \qquad \left(\int_{-1}^{1}\right)^{n} (x^{i})^{k-1}\, C_m^n(X)\, dX = 0$$

for $1 \leq k \leq m$ it is easy to show that the coefficients $a_1, \ldots, a_m$ of $C_{2m}^n(X)$ can be obtained by solving the matrix equation

$$(12) \qquad \begin{bmatrix} \frac{1}{3} & \frac{1}{5} & \cdots & \frac{1}{2m+1} \\ \frac{1}{5} & \frac{1}{7} & \cdots & \frac{1}{2m+3} \\ \vdots & & \ddots & \\ \frac{1}{2k+1} & \frac{1}{2k+3} & \cdots & \frac{1}{2m+2k-1} \\ \vdots & & \ddots & \\ \frac{1}{2m+1} & \frac{1}{2m+3} & \cdots & \frac{1}{4m-1} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \\ \vdots \\ a_m \end{bmatrix} = -\frac{1}{n} \begin{bmatrix} 1 \\ \frac{1}{3} \\ \vdots \\ \frac{1}{2k-1} \\ \vdots \\ \frac{1}{2m-1} \end{bmatrix}$$

Similarly, the coefficients $a_1, \ldots, a_m$ of $C_{2m+1}^n(X)$ can be obtained by solving the matrix equation

$$(13) \qquad \begin{bmatrix} \frac{1}{5} & \frac{1}{7} & \cdots & \frac{1}{2m+3} \\ \frac{1}{7} & \frac{1}{9} & \cdots & \frac{1}{2m+5} \\ \vdots & & \ddots & \\ \frac{1}{2k+3} & \frac{1}{2k+5} & \cdots & \frac{1}{2m+2k+1} \\ \vdots & & \ddots & \\ \frac{1}{2m+3} & \frac{1}{2m+5} & \cdots & \frac{1}{4m+1} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \\ \vdots \\ a_m \end{bmatrix} = -1 \begin{bmatrix} \frac{1}{3} \\ \frac{1}{5} \\ \vdots \\ \frac{1}{2k+1} \\ \vdots \\ \frac{1}{2m+1} \end{bmatrix}$$

Considering the case $n = 1$ in equations (12) and (13) the polynomials $C_m^n(X)$ are readily identified with the Legendre Polynomials by the equation

$$(14) \qquad C_m^n(X) = a_m \sum_{i=1}^{n} P_m(x^i)$$

where a_m is a constant of proportionality.

It is, moreover, easy to see that for $m > 0$,

$$(15) \qquad \left(\int_{1}^{1}\right)^{n} C_m^n(X) \prod_{i=1}^{n} (x^i)^{k_i}\, dx^i = 0$$

whenever each integer k_i satisfies $0 \leqq k_i < m$.

ii) <u>The choice of zeros of the polynomials</u>

For $m > 0$, $n \geqq 1$, the equation $C_m^n(X) = 0$ has an infinite number of roots. Moreover, constructing interpolation polynomials using $C_m^n(X)$ presents extreme difficulties. If, however, we chose the zeros in a manner corresponding to the way we have set up the non-linear algebraic equations; that is by first setting all variables except one equal to zero and then equating the resulting polynomial in one variable to zero, we find these zeros to be the required non-linear unknowns in our simultaneous non-linear algebraic equations. For example the zeros of $C_2^n(X)$ chosen in this way are $u = \pm (n/3)^{1/2}$, those of $C_3^n(X)$ are $u = 0$, $\pm (3/5)^{1/2}$, and those of $C_5^n(X)$ are $u,v = 0, \pm (\frac{5}{9}[1 \pm (\frac{8}{35})^{1/2}])^{1/2}$ these being the solutions to the non-linear algebraic equations (7), (8) and (9) respectively for the n-cube.

The zeros of $C_{2m+1}^n(X)$ chosen as above will always be the zeros of $P_{2m+1}(X)$. The zeros of $C_{2m}^n(X)$ are unfortunately complex for $m > 1$ -- it can, moreover, be shown that the complex zeros of $C_4^n(X)$ are not the solution to the non-linear algebraic equations set up in the manner described above for obtaining an integration formula of precision 7.

Hence only the polynomials $C_{2m+1}^n(X)$ are useful for obtaining numerical integration formulas of precision $p = 4k + 1$ for $n \geqq (p-1)/2$.

iii) <u>Polynomials orthogonal over an arbitrary fully symmetric region in E^n</u>

Since our even polynomials were not useful for the specific case, the n-cube, a set of even polynomials will not be useful in the general case.

The odd ones, however, prove to be helpful. We define these to be

$$(16)\qquad Q^n_{2m+1}(X) = \sum_{i=1}^{n} \left(x^i + \sum_{j=1}^{m} a_j (x^i)^{2j+1}\right) = \sum_{i=1}^{n} q_{2m+1}(x^i)$$

where the a_j's can be obtained by solving a system of m simultaneous linear algebraic equations, the k'th of which may be written

$$(17)\qquad \sum_{j=1}^{m} a_j \int_{R_n} w(X)(x^i)^{2(j+k+1)} dX = -\int_{R_n} w(X)(x^i)^{2k}\, dX\ ,$$

R_n and $w(X)$ being fully symmetric as described above.

Although these polynomials are orthogonal over R_n to any monomial of the form $(x^i)^{k_i}$, they are not, in general, orthogonal to any monomial of the form $\prod_{i=1}^{n}(x^i)^{k_i}$ where $0 \leqq \sum_{i=1}^{n} k_i \leqq 2m$.

The zeros of $Q^n_{2m+1}(X)$ chosen in the manner described above are obviously those of $q_{2m+1}(x)$. Considering the properties of R_n and $w(X)$ we may write*, with the x_{m_j}'s being the zeros of $q_{2m+1}(x)$,

$$\int_{R_n} w(X) \prod_{j=1}^{k} (x^i - x^i_{m_j})\, q_{2m+1}(x^i)\, dX$$

$$= \int_{R_n} w(X) \prod_{j=1}^{k} (x^n - x^n_{m_j})\, q_{2m+1}(x^n)\, dX$$

$$= \int_{-\alpha}^{\alpha} \int_{-\theta_2(x^n)}^{\theta_2(x^n)} \cdots \int_{-\theta_n(x^2,\ldots,x^n)}^{\theta_n(x^2,\ldots,x^n)} w(X) \prod_{j=1}^{k} (x^n - x^n_{m_j}) q_{2m+1}(x^n)\, dX$$

Due to the symmetry of the region R_n, the functions $\theta_2(x^n),\ldots,$ $\theta_n(x^2,\ldots,x^n)$ are all even functions of their independent variables. If, in addition we assume these functions positive throughout R_n (but not necessarily on the boundary) we may write

---

* We assume here that R_n contains the origin, and that the limits of integration can be written as we have written them.

$$\int_{R_n} w(X) \prod_{j=1}^{k} (x^i - x^i_{m_j}) q_{2m+1}(x^i)\, dX = \int_{-\alpha}^{\alpha} \omega(x^2) \prod_{j=1}^{k} (x - x_{m_j}) q_{2m+1}(x)\, dx$$

where $\omega(x^2)$ is positive throughout the interval $(-\alpha,\alpha)$. Hence by well known arguments (see, for example [13]) all the zeros of $q_{2m+1}(x)$ are real, distinct, and lie in the open interval $-\alpha < x < \alpha$.

Again, by well-known arguments, the zeros of the polynomial $q_{2m+1}(x)$ can be shown to be the non-linear unknowns in the non-linear algebraic equations set up in the manner we have set them up (e.g. for the cases $p = 5$, $p = 9$) for obtaining integration formulas of precision $p = 4m + 1$, $n \geqq (p-1)/2$. Our reasons for this statement are as follows: Assuming $2m+1$ distinct non-linear unknowns (including zero) we can always obtain from these combinations of fully symmetric sets of points which will enable us to integrate all monomials up to degree $4m+1$ (although we have not yet found any examples for which equation (10) does not hold, restrictive conditions such as these may prevent us from obtaining solutions in all cases). The $2m$ equations resulting from integrating $(x^i)^2$, $(x^i)^4$, ..., $(x^i)^{2m}$ uniquely determine all our non-linear unknowns (for proof see [14]). We can then solve the resulting system of linear algebraic equations.

For example, if we multiply equation (ii) of (9) by u^2, subtract (iii) from it, call the result a, (iii) of (9) by u^2, subtract (iv) from it, call the result b, (iv) of (9) by u^2, subtract (v) from it, call the result c, then $v^2 = a/b = b/c$ is a polynomial equation whose solution is (11), these roots being the same as those of $q_5(x)$ (excluding the zero root). This, then, is a partial proof of the validity of the solution to the algebraic equations (9). We may easily obtain the value of F by subtracting equation (viii) from (ix). By successively eliminating the unknowns I, J, ... from equations (vi), (vii), (viii), (x), (xi), (xii) we can solve

for D, E, G, H, I, J; a check being that $I_{62} + u^2v^2I_{22} = (u^2+v^2)I_{42}$. Any two of equations (ii) and (v) will give the solution to B and C. From equation (i) we can then obtain the value of A.

In carrying out this outlined procedure we would prove the validity of the given solution to the non-linear algebraic equations (9).

iv) Polynomials orthogonal over the n-sphere

For completeness we also find the polynomials $Q^n_{2m+1}(X) = S^n_{2m+1}(X)$ orthogonal over the n-sphere with respect to the weight function 1.

We transform the integral over the n-sphere to an integral over an n dimensional parallelepipedon using polar coordinates. Now for the circle (2-sphere) the transformation that will do this for us is

$$(18) \qquad \begin{cases} x^1 = r\cos\theta^1 \\ x^2 = r\sin\theta^1 \end{cases}$$

Similarly for the sphere (3-sphere) we may let

$$(19) \qquad \begin{cases} x^1 = r\cos\theta^1 \\ x^2 = r\sin\theta^1\cos\theta^2 \\ x^3 = r\sin\theta^1\sin\theta^2 \end{cases}$$

to transform an integral over a sphere to an integral over a rectangular region in three dimension. Both (18) and (19) satisfy

$$(20) \qquad \sum_{i=1}^{n} (x^i)^2 = r^2$$

and hence using (20) in view of (18) and (19) the transformation

$$
(21) \quad \begin{cases}
x^1 = r\cos\theta^1 \\
x^2 = r\sin\theta^1\cos\theta^2 \\
x^3 = r\sin\theta^1\sin\theta^2\cos\theta^3 \\
\quad\cdot \\
\quad\quad\cdot \\
\quad\quad\quad\cdot \\
x^{n-2} = r\sin\theta^1\sin\theta^2\ldots\sin\theta^{n-3}\cos\theta^{n-2} \\
x^{n-1} = r\sin\theta^1\sin\theta^2\ldots\sin\theta^{n-3}\sin\theta^{n-2}\cos\theta^{n-1} \\
x^n \;\;= r\sin\theta^1\sin\theta^2\ldots\sin\theta^{n-3}\sin\theta^{n-2}\sin\theta^{n-1}
\end{cases}
$$

extends (18) and (19) to the n-sphere, where $n \geqq 2$.

We next proceed to find the jacobian $\mathcal{J}_n$ of the transformation (21). Here it becomes convenient to define the more concise notations

$$
(22) \quad \begin{cases}
\mu^k = \cos\theta^k \\
\nu^k = \sin\theta^k \\
J_n = \mathcal{J}_n / r^{n-1}
\end{cases} .
$$

Then from the well-known definition of the jacobian $= |a_{ij}|$ where on transforming from the variables x to the variables $\bar{x}$, $a_{ij} = \dfrac{\partial x^i}{\partial \bar{x}^j}$, it is easily shown that in terms of our definition (22) and equation (21)

(23) $\quad J_n =$ (see page 57).

On inspecting the last two rows of this determinant we note that

$$
\begin{aligned}
J_n &= (\mu^{n-1})^2 \prod_{i=1}^{n-2} (\nu^i) J_{n-1} + (\nu^{n-1})^2 \prod_{i=1}^{n-2} (\nu^i) J_{n-1} \\
&= ((\mu^{n-1})^2 + (\nu^{n-1})^2) J_{n-1} \prod_{i=1}^{n-2} \nu^i , \qquad \text{or}
\end{aligned}
$$

$$
(24) \quad J_n = J_{n-1} \prod_{i=1}^{n-2} \nu^i
$$

$$
(23)\quad J_n = \left| \begin{array}{ccccccc}
\mu^1 & -\nu^1 & 0 & \cdots & 0 & 0 & 0 \\
\mu^2\nu^1 & \frac{\mu^1}{\nu^1}\mu^2\nu^1 & -\nu^1\nu^2 & \cdots & 0 & 0 & 0 \\
\mu^3\nu^1\nu^2 & \frac{\mu^1}{\nu^1}\mu^3\nu^1\nu^2 & \frac{\mu^2}{\nu^2}\mu^3\nu^1\nu^2 & \cdots & 0 & 0 & 0 \\
\hdashline
\mu^{n-2}\prod_{i=1}^{n-3}\nu^i & \frac{\mu^1}{\nu^1}\mu^{n-2}\prod_{i=1}^{n-3}\nu^i & \frac{\mu^2}{\nu^2}\mu^{n-2}\prod_{i=1}^{n-3}\nu^i & \cdots & \frac{\mu^{n-3}}{\nu^{n-3}}\mu^{n-2}\prod_{i=1}^{n-3}\nu^i & -\prod_{i=1}^{n-2}\nu^i & 0 \\
\mu^{n-1}\prod_{i=1}^{n-2}\nu^i & \frac{\mu^1}{\nu^1}\mu^{n-1}\prod_{i=1}^{n-2}\nu^i & \frac{\mu^2}{\nu^2}\mu^{n-1}\prod_{i=1}^{n-2}\nu^i & \cdots & \frac{\mu^{n-3}}{\nu^{n-3}}\mu^{n-1}\prod_{i=1}^{n-2}\nu^i & \frac{\mu^{n-2}}{\nu^{n-2}}\mu^{n-1}\prod_{i=1}^{n-2}\nu^i & -\prod_{i=1}^{n-1}\nu^i \\
\prod_{i=1}^{n-1}\nu^i & \frac{\mu^1}{\nu^1}\prod_{i=1}^{n-1}\nu^i & \frac{\mu^2}{\nu^2}\prod_{i=1}^{n-1}\nu^i & \cdots & \frac{\mu^{n-3}}{\nu^{n-3}}\prod_{i=1}^{n-1}\nu^i & \frac{\mu^{n-2}}{\nu^{n-2}}\prod_{i=1}^{n-1}\nu^i & \frac{\mu^{n-1}}{\nu^{n-1}}\prod_{i=1}^{n-1}\nu^i
\end{array} \right|
$$

since $(\mu^k)^2 + (\nu^k)^2 = 1$. Since $J_2 = 1$, $J_3 = \nu^1$, we have $J_n = (\nu^1)^{n-2}(\nu^2)^{n-3}\ldots(\nu^{n-3})^2(\nu^{n-2})$, from which we deduce that

$$(25) \qquad \mathcal{J}_n = r^{n-1} \prod_{i=1}^{n-2} (\sin\theta^{n-i-1})^i \quad .$$

For purposes of integration over the n-dimensional unit sphere the ranges of variables may be chosen to be

$$(26) \qquad \begin{aligned} &0 \leqq r \leqq 1 \\ &0 \leqq \theta^i \leqq \pi \qquad i = 1,\ldots,n-2 \\ &0 \leqq \theta^{n-1} \leqq 2\pi \end{aligned}$$

and we have now arrived at the result that

$$(27) \qquad \int\cdots\int f(x^1,\ldots,x^n)dx^1\ldots dx^n$$

$$= \int_0^{2\pi}\left(\int_0^{\pi}\right)^{n-2}\int_0^1 F(r,\theta^1,\ldots,\theta^{n-1})r^{n-1}dr \prod_{i=1}^{n-2}((\sin\theta^{n-i-1})^i d\theta^i)d\theta^{n-1}$$

where $F(r,\theta^1,\ldots,\theta^{n-1}) = f(r\cos\theta^1,\ldots,r\sin\theta^1\sin\theta^2\ldots\sin\theta^{n-1})$ under the transformation (21).

In terms of the transformation (21) and the notations (22) we have

$$(28) \qquad \prod_{i=1}^{n} (x^i)^{k_i} = r^{\sum_{i=1}^{n}k_i}(\mu^1)^{k_1}(\nu^1\mu^2)^{k_2}\ldots(\nu^1\nu^2\ldots\nu^{n-2}\mu^{n-1})^{k_{n-1}} \cdot (\nu^1\nu^2\ldots\nu^{n-1})^{k_n}$$

$$= r^{\sum_{i=1}^{n}k_i} \prod_{i=1}^{n-1} ((\mu^i)^{k_i}(\nu^i)^{\sum_{s=i+1}^{n}k_s})$$

Substituting k for $n-i-1$ in (25) and writing i for k yields

$$(29) \qquad J_n = \prod_{i=1}^{n-2} (\nu^i)^{n-i-1} = \prod_{i=1}^{n-1} (\nu^i)^{n-i-1} \quad .$$

On multiplying the right hand side of (28) by the right hand side of (29) and writing $\cos\theta^i$ for μ^i, $\sin\theta^i$ for ν^i we obtain the integration formula

$$(30)\qquad \underset{n\text{-sphere}}{\int\cdots\int} \prod_{i=1}^{n} (x^i)^{k_i}\, dx^i$$

$$= \int_0^{2\pi}\left(\int_0^{\pi}\right)^{n-2}\int_0^1 r^{-1+\sum_{i=1}^{n}(k_i+1)}\, dr \prod_{i=1}^{n-1}\left((\cos\theta^i)^{k_i}(\sin\theta^i)^{n-i-1+\sum_{s=i+1}^{n}k_s} d\theta^i\right)$$

If at least one of the k_i's is an odd integer the integral (30) is zero due to the symmetry of the region of integration. If, however all the k_i's are even positive integers or zero, (30) will be different from zero. To evaluate (30) we borrow a well-known result from the theory of the Gamma function, namely

$$(31)\qquad \int_0^{\frac{\pi}{2}} (\sin\theta)^j(\cos\theta)^k d\theta = \frac{1}{2}\,\frac{\Gamma(\frac{k+1}{2})\,\Gamma(\frac{j+1}{2})}{\Gamma\,(\frac{j+k+2}{2})}$$

where in our applications we assume j and k are zero or positive integers.

Writing $2k_i$ for k_i in (30) and using (31) we obtain

$$(32)\qquad \underset{n\text{-sphere}}{\int\cdots\int} \prod_{i=1}^{n} (x^i)^{2k_i}\, dx^i$$

$$= \frac{2}{\sum_{i=1}^{n}(2k_i+1)} \prod_{i=1}^{n-1} \frac{\Gamma(\frac{2k_i+1}{2})\,\Gamma(\frac{n-1+\sum_{s=i+1}^{n}2k_s}{2})}{\Gamma\,(\frac{2k_i+n-i+\sum_{s=i+1}^{n}2k_s}{2})} .$$

Due to cancellation of successive terms in the above product, (32) reduces to

$$(33)\qquad \underset{n\text{-sphere}}{\int\cdots\int} \prod_{i=1}^{n} (x^i)^{2k_i}\, dx^i = \frac{2\prod_{i=1}^{n}\Gamma(k_i+\frac{1}{2})}{(\sum_{i=1}^{n}(2k_i+1))\,(\Gamma(\frac{n}{2}+\sum_{i=1}^{n}k_i))}$$

which is in agreement (by putting each $k_i = 0$) with the well-known result for the hypervolume of the n-sphere*

$$(34) \qquad V_n = \int \cdots \int_{n\text{-sphere}} \prod_{i=1}^{n} dx^i = \frac{\pi^{n/2}}{\Gamma(1+\frac{n}{2})}$$

Using formula (34) together with the notation $(a)_k = a(a+1)\dots(a+k-1)$, $(a)_0 = 1$, and the recurrence relation $\Gamma(1+x) = x\Gamma(x)$ for the Gamma function we finally obtain

$$(35) \qquad \int \cdots \int_{n\text{-sphere}} \prod_{i=1}^{n} (x^i)^{2k_i} dx^i = \frac{\prod_{i=1}^{n} (\frac{1}{2})_{k_i}}{(\frac{n+2}{2})_{(\sum_{i=1}^{n} k_i)}} \cdot V_n$$

We can now write down the set of linear algebraic equations from which we can find the values of the coefficients $a_1, \dots, a_m$ of $Q^n_{2m+1}(X) = S^n_{2m+1}(X)$. For on substituting the appropriate form of (35) into (17) and simplifying, we find these equations to be

$$(36) \qquad \sum_{s=1}^{m} \frac{(\frac{2r+1}{2})_s}{(\frac{n+2r+2}{2})_s} . a_s = -1, \quad r = 1 \text{ to } m.$$

Equations (35) and (36) can be used to:

(i) find the explicit solution to equations (9)

(ii) verify the solution to equations (8)

(iii) verify that (10) holds

(iv) find numerical integration formulas of higher precisions for the n-sphere.

If we write $S^n_{2m+1}(X) = \sum_{i=1}^{n} s_{2m+1}(x^i)$, then according to the result of pages 53-54 all the zeros of $s_{2m+1}(x)$ lie in the open interval $-1 < x < 1$.

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* We have used the result $\Gamma(\frac{1}{2}) = \sqrt{\pi}$.

CHAPTER IV

INTEGRATION FORMULAS OF ARBITRARY PRECISION

In Chapter III we have developed a relatively simple method of obtaining numerical integration formulas by reducing the solution of a system of non-linear algebraic equations to the solution of a system of linear algebraic equations. The formulas we can obtain by this procedure are of precision p over fully symmetric regions in n-space where $p = 4k + 1$, k an integer. To overcome the restriction this equation imposes on p we call on the repeated Gaussian formulas, noting (see Table II for example) that for large p and moderate n these formulas are not unduly extravagant in ordinate evaluation, particularly since they yield greater accuracy for larger values of p.

Arbitrary limits of integration in an n-fold integral are somewhat inconvenient and the practical task of evaluation is eased if we can transform the limits to $(-1,1)$, i.e. if we can transform an integral into an integral over the n-cube. A transformation is given which enables us to do this.

In this chapter, formulas of arbitrarily high precision over the finite and infinite n-sphere are constructed. These formulas are included for completeness rather than for practical value. In practice, we are restricted to values of p and n which can be encompassed in a reasonable time on existing hardware. The formulas can however achieve remarkable accuracy in evaluating integrals.

1) Formulas over Rectangular Regions

The methods of Chapter III suffice for formulas of precision $p = 4k + 1$, k an integer.

It is desirable that formulas of all odd precisions be available, particularly for moderate values of n. The main difficulty impeding the development of economical high precision formulas is the brute difficulty of solving a large system of non-linear equations in a number of unknowns. No general methods are available and a search of the literature suggests that little research is being done in non-linear systems. In this Chapter we evade the difficulty by using Gauss-product formulas.

Suppose the region of integration is a rectangular region in n-space. The Gaussian integration formulas produce very good accuracy for

the large class of functions that have all their derivatives continuous in this region of integration. Moreover the number of evaluation points required is $(\frac{p+1}{2})^n$ *; for fixed n the number of points increases polynomially in p, not exponentially. Thus if the number of dimensions in n-space is small (e.g. 2, 3, 4, 5) we may be able to obtain accurate results with a feasible number of points.

The construction of product-type integration formulas over rectangular regions in n-space is not difficult. Let the region of integration be described by

$$a^i \leqq x^i \leqq b^i \qquad **$$

and let the weight function $w(X)$ have the form

$$w(X) = \prod_{i=1}^{n} w_i(x^i) \quad ,$$

$w_i(x^i) > 0$ in $a^i < x^i < b^i$; then if we can find polynomials orthogonal over the intervals (a^i, b^i) with respect to the corresponding weight functions $w_i(x^i)$ we can find the Gaussian integration formulas of arbitrary precision over the intervals (a^i, b^i) with respect to the weight functions $w_i(x^i)$. Using the Cartesian product theorem given on page 11 we easily obtain numerical integration formulas of arbitrary precision over

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* We actually have a variation here; more generally we require $\prod_{i=1}^{n} (\frac{p_i+1}{2})$ where p_i is the precision obtained in each variable.

** Here a^i and b^i may be finite or infinite, assuming the weight functions $w_i(x^i)$ to be appropriately chosen.

rectangular regions in n-space.

If it is not possible to write $w(X) = \prod_{i=1}^{n} w_i(x^i)$, or if some of the $w_i(x^i) \not> 0$ everywhere in (a^i, b^i), we can always include such weight functions as part of our integrand and apply standard Gaussian formulas (Legendre, Laguerre and Hermite); e.g. if we define $w(X)\ f(X) = F(X)$, we have

$$(1) \qquad \int_{a^1}^{b^1} \cdots \int_{a^n}^{b^n} w(X)\ f(X)\ dX = \int_{a^1}^{b^1} \cdots \int_{a^n}^{b^n} F(X)\ dX \quad .$$

For example if the limits of integration are finite we know that we can employ the Gauss-Legendre formulas to evaluate the above integral on the right to arbitrary precision*. Due to the additional non-linearity of the weight function $w(X)$ we will not in general be integrating $f(X)$ to precisions p_i (see footnote on previous page) when integrating $F(X)$ to precisions p_i .

2) Transformation from an Arbitrary Region to the n-Cube

The results of this section make it possible to apply the repeated Gaussian integration formulas to integrals of the type (2) below. The following theorem enables us to carry out the transformation.

THEOREM. Given the integral

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* If some of the limits of integration are infinite we can employ Gauss-Laguerre integration (and similarly, Gauss-Hermite) by setting $F(\cdot) = \exp(-x^i)\ G(\cdot)$; $G(\cdot) = \exp(x^i)\ F(\cdot)$.

$$(2) \qquad I = \int_{\varphi^1}^{\psi^1} \int_{\varphi^2(x^1)}^{\psi^2(x^1)} \cdots \int_{\varphi^n(x^1,\ldots,x^{n-1})}^{\psi^n(x^1,\ldots,x^{n-1})} f(x^1,x^2,\ldots,x^n)dx^n dx^{n-1}\ldots dx^1$$

we can always transform this into an integral over the n-cube by a sequence of n linear transformations.

PROOF. The symmetric case $\varphi^i = -\psi^i$, $i = 1,\ldots,n$ (φ^1 and ψ^1 are constants) is easily tractable and we consider it first. We set

$$(3) \qquad x^i = \psi^i u^i \qquad i = 1,\ldots,n \ .$$

Then

$$(4) \qquad \begin{bmatrix} dx^1 \\ dx^2 \\ \cdot \\ \cdot \\ \cdot \\ dx^n \end{bmatrix} = \begin{bmatrix} \psi^1 & 0 & \cdots & 0 \\ u^2\psi^2_1 & \psi^2 & \cdots & 0 \\ & \ddots & & \\ u^n\psi^n_1 & u^n\psi^n_2 & \cdots & \psi^n \end{bmatrix} \begin{bmatrix} du^1 \\ du^2 \\ \cdot \\ \cdot \\ \cdot \\ du^n \end{bmatrix}$$

where by ψ^i_j we mean $\dfrac{\partial\psi^i}{\partial u^j}$. From (4), the jacobian J of the transformation (3) is just the product of the diagonal elements of the matrix:

$$(5) \qquad J = \prod_{i=1}^{n} \psi^i \ .$$

Hence we have

$$(6) \qquad \begin{aligned} &\int_{-\psi^1}^{\psi^1} \cdots \int_{-\psi^n(x^1,\ldots,x^{n-1})}^{\psi^n(x^1,\ldots,x^{n-1})} f(x^1,\ldots,x^n)\, dx^n \ldots dx^1 \\ &= \int_{-1}^{1} \cdots \int_{-1}^{1} F(u^1,\ldots,u^n)\, J\, du^n \ldots du^1 \end{aligned}$$

where J is given by (5), and $F(u^1,\ldots,u^n) = f(x^1,\ldots,x^n)$ under the transformation (3).

If the limits of integration are as in (2) then the set of transformations

(7) $$x^i = \frac{\psi^i + \varphi^i}{2} + u^i \frac{(\psi^i - \varphi^i)}{2} \qquad i = 1,\dots,n$$

will transform the integral (2) to an integral over the n-cube, and the proof is similar to that for the symmetric case above. The jacobian of the transformation (7) is

(8) $$J = \prod_{i=1}^{n} [\frac{\psi^i - \varphi^i}{2}] \quad .$$

This theorem thus allows us to apply Gaussian integration formulas to integrals of the type (2). It may be worth while finding integration formulas with the functions J as weight functions if J is suitable and the value of integrals are required over the region for a sufficient number of functions $f(\cdot)$. If we require the value of only one integral it will likely be more economical of programming time to use standard Gaussian integration formulas (e.g. Legendre, Chebychev) applicable over the interval $(-1,1)$ with more points to sustain accuracy. Considerable extra expense of machine time is tolerable for one-shot evaluates.

Let us consider the simple case in which the function $f(\cdot)$ is a polynomial in n variables $x^i(i=1,\dots,n)$. Then $F(\cdot) = f(\cdot)$ under the transformation (7) will in general not be a polynomial in the variables $u^i(i = 1,\dots,n-1)$ although it will be a polynomial in u^n of the same degree as $f(\cdot)$ was in x^n. Hence the transformation (7) will introduce complications in $n-1$ of the variables of the integrand and when performing numerical integration the precision obtained may not be as high in the original variables as in the new variables.

As an example, let us transform the integral over the circle

$$I = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\, dy\, dx$$

onto the corresponding integral over the square. By equation (4) the linear transformations which will enable us to do this are $x = u$, $y = v\sqrt{1-x^2} = v\sqrt{1-u^2}$. By equation (5) the jacobian of the transformation is $\sqrt{1-u^2}$. Hence

$$I = \int_{-1}^{1}\int_{-1}^{1} f(u, v\sqrt{1-u^2})\,\sqrt{1-u^2}\,dv\,du \quad .$$

We can now perform repeated Gaussian integration to evaluate this integral; Gauss-Lengendre in the variable v, Gaussian-Chebychev* in the variable u. The resulting integrand $f(u, v\sqrt{1-u^2})$ is however, more complicated in u than $f(x,y)$ was in x (even though the Gauss-Legendre integration will remove all odd powers of $\sqrt{1-u^2}$ in $f(u, v\sqrt{1-u^2})$.

This transformation theorem is of course limited in scope to integrals of the type employed in the statement of the theorem. Within this limitation, the theorem is of practical value since it enables us to prescind from the task of finding integration formulas valid for arbitrary (bounded) regions in n-space and to concentrate on the simpler problem of finding higher precision integration formulas for the n-cube, keeping the weight function arbitrary.

3) Formulas of Arbitrary Precision over the n-Sphere

This problem has already been solved by W. H. Peirce for the cases $n = 2$ and $n = 3$ (see [15] and [16]). The formulas developed (here) in

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* We can employ the integration formula

$$\int_{-1}^{1}\sqrt{1-x^2}\,g(x)\,dx \cong \sum_{j=1}^{m} b_j\,g(x_j)\ ;\quad b_j = \frac{\pi \sin^2[\frac{j\,\pi}{m+1}]}{m+1}\ ,\quad x_j = \cos(\frac{j\,\pi}{m+1})\ .$$

The x_j's are zeros of the Chebychev polynomial of the second kind, $U_n(x)$, orthogonal over the interval $(-1,1)$ with respect to the weight function $\sqrt{1-x^2}$.

section 4 of the last Chapter enable us to extend the results of W. H. Peirce to the n-sphere, where $n \geqq 2$.

We shall seek numerical integration formulas of a fairly general form, namely

$$(9) \qquad \underset{n\text{-sphere}}{\int \cdots \int} w\left(\sqrt{\sum_{i=1}^{n} (x^i)^2}\right) f(x^1,\ldots,x^n)\,dx^1 dx^2 \ldots dx^n$$

$$\cong \sum_{j_0=1}^{m_0} \sum_{j_1=1}^{m_1} \cdots \sum_{j_{n-1}=1}^{m_{n-1}} a_{j_0 j_1 \ldots j_{n-1}} f(x^1_{j_0 j_1 \ldots j_{n-1}},\ldots,x^n_{j_0 j_1 \ldots j_{n-1}}) ;$$

the radius of the sphere may be finite or infinite depending of the weight function $w(\cdot)$.

By formulas (27) and (29) of Chapter III we may write

$$(10) \qquad \underset{n\text{-sphere}}{\int \cdots \int} w(\cdot)\, f(x^1,\ldots,x^n)\,dx^1 \ldots dx^n$$

$$= \int_0^R w(r) r^{n-1} \prod_{i=1}^{n-2} \left(\int_0^{\pi} (\sin\theta^i)^{n-i-1} \right) \int_0^{2\pi} F(r,\theta^1,\theta^2,\ldots,\theta^{n-1}) \cdot$$

$$\cdot\, d\theta^{n-1} \ldots d\theta^1 dr$$

where

$$F(r,\theta^1,\ldots,\theta^{n-1}) = f(r\cos\theta^1,\ldots,r\sin\theta^1\ldots\sin\theta^{n-1})$$

under the transformation (21) of Chapter III. For $n = 2$ we exclude the product $\prod(\int \cdot)$ in formula (10).

Using the Cartesian product theorem (page 11) and the transformation theorem (page 9) we know that we can find the integration formulas (9) if we find the integration formulas

$$(11) \qquad \int_0^{2\pi} f_{n-1}(\theta)\, d\theta \cong \sum_{j_{n-1}=1}^{m_{n-1}} c^{n-1}_{j_{n-1}} f_{n-1}(\theta_{j_{n-1}})$$

$$(12) \qquad \int_0^{\pi} (\sin\theta)^{n-k-1} f_k(\theta)d\theta \cong \sum_{j_k=1}^{m_k} c^k_{j_k} f_k(\theta_{j_k}) \;^{*} \qquad k = 1 \text{ to } n-2$$

$$(13) \qquad \int_0^{R} w(r)\, r^{n-1} f_o(r)dr \cong \sum_{j_o=1}^{m_o} c^o_{j_o} f_o(r_{j_o}) \; .$$

Moreover, in view of the transformation (21) of Chapter III, if the formulas (13), (12) and (11) will be of precision p in the variables r, $\sin\theta$ and $\cos\theta$, formula (9) will in turn be of precision p in the variables x^i.

Putting $y = \cos(\theta/2)$ in (11) we obtain

$$(14) \qquad \int_0^{2\pi} f_{n-1}(\theta)d\theta = 2\int_{-1}^{1} \frac{f_{n-1}(2\cos^{-1}y)}{(1-y^2)^{\frac{1}{2}}}\,dy \cong \sum_{j_{n-1}=1}^{m_{n-1}} c^{n-1}_{j_{n-1}} f(2\cos^{-1}y_{j_{n-1}})$$

this being the well-known Chebychev integration formula (formula (8) Chapter I). Chosing $m_{n-1} = 2m$ we obtain precision $p = 4m-1$ in the variable $y = \cos(\theta/2)$; or $p = 2m-1$ in the variable y^2 -- in this case we would also obtain precision $p = 2m-1$ in the variables $\cos\theta$ and $\sin\theta$. The zeros y_j and weights c_j for formula (14) are given in Table VI.

Letting $y = \cos\theta$ in (12), we obtain

$$(15) \qquad \int_0^{\pi} (\sin\theta)^{n-k-1} f_k(\theta)d\theta = \int_{-1}^{1} (1-y^2)^{(n-k-2)/2} f_k(\cos^{-1}y)dy$$

$$\cong \sum_{j_k=1}^{m_k} c^k_{j_k} f(\cos^{-1}y_{j_k}) \; .$$

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* For $n = 2$ these formulas are missing.

Here we assume $n > 2$, $k = 1$ to $n-2$. If we take $m_k = m$ in (15) and use the zeros and weights as indicated in Table VI we obtain precision $p = 2m-1$ in $y = \cos\theta$, and on inspecting the transformation (21) of Chapter III we note that the errors in the odd powers of $\sin\theta$ in formula (15) will cancel out in our final integration formula (9) due to all integration points being spaced symmetrically about $\theta = \pi/2$.

We consider three cases of the integration formulas (13):

$$\begin{aligned} &(i) & w(r) &= 1, & R &= 1; \\ &(ii) & w(r) &= e^{-r}, & R &= \infty; \\ &(iii) & w(r) &= e^{-r^2}, & R &= \infty. \end{aligned}$$

We assume that $n \geqq 2$.

In the cases (i) and (ii) the zeros and weights required are readily obtained from the references indicated in Table VI. With m points in formula (13) we obtain polynomial precision $2m-1$ in r. Case (iii) is readily identified with case (ii) by setting $y = r^2$.

We have used the notation of [18] in Table VI.

Hence we have found the integration formulas of arbitrary precision over the finite and infinite n-sphere. Formula (9) may now be written

$$(16) \qquad \underset{n\text{-sphere}}{\int \cdots \int} w\left(\sqrt{\sum_{i=1}^{n} (x^i)^2}\right) f(x^1, \ldots, x^n)\, dx^1 \ldots dx^n$$

$$\cong \sum_{j_0=1}^{m} \sum_{j_1=1}^{m} \cdots \sum_{j_{n-2}=1}^{m} \sum_{j_{n-1}=1}^{2m} c^0_{j_0}\, c^1_{j_1} \cdots c^{n-1}_{j_{n-1}}\, f(x^1_{j_0 \ldots j_{n-1}}, \ldots, x^n_{j_0 \ldots j_{n-1}})$$

TABLE VI

ZEROS AND CHRISTOFFEL NUMBERS OF ORTHOGONAL POLYNOMIALS

INTERVAL	WEIGHT FUNCTION	ORTHOGONAL POLYNOMIAL	ZEROS	WEIGHTS (CHRISTOFFEL NUMBERS)
$(-1,1)$	$1/\sqrt{1-x^2}$	$T_m(x) = \cos m\theta$; $x = \cos\theta$	$\cos\left[\frac{(2j-1)\pi}{2m}\right]$	$\frac{\pi}{m}$ See [29]
$(-1,1)$	$(1-x^2)^\alpha$	$P_m^{(\alpha,\alpha)}(x)$ $=2^{-m}\sum_{s=0}^{m}\binom{m+\alpha}{s}\binom{m+\alpha}{m-s}(x-1)^s$	x_j	$\frac{(1-x_j^2)[\Gamma(m+\alpha)P_m^{(\alpha,\alpha)}(x_j)]^2}{2^{-\alpha-1}m!\,\Gamma(m+2\alpha+1)}$ See [29, 30, 31]
$(0,1)$	r^{n-1}	$P_m^{(0,n-1)}(2r-1)$ $=2^{-(n-1)}\sum_{s=0}^{m}\binom{m}{s}\binom{m+n-1}{m-s}r^s$ $(r-1)^s$	r_j	See [17]
$(0,\infty)$	$x^\alpha e^{-x}$	$L_m^\alpha(r)$ $=\sum_{s=0}^{m}\binom{m+\alpha}{m-s}\frac{(-r)^s}{s!}$	See [28,32]	$\frac{\Gamma(m+\alpha+1)}{(m+1)!\,L_{m-1}^{\alpha+1}(x_j)L_{m+1}^{\alpha}(x_j)}$

where the constants $c^k_{j_k}$, $k = 0$ to $n - 1$ can be obtained from Table VI. The points $x_{j_0 \cdots j_{n-1}}$ can be obtained from the references indicated in Table VI together with the formulas of the transformation (21) of Chapter III.

We see that the total number of points required to obtain this precision $(2m-1)$ is $2(m)^n$. This number of points (for moderate p and large n) is much too large since by the method of the previous chapter, formulas of precision p require only $O[n^{(p-1)/2}]$ points. However, the formulas of this Chapter are not too extravagant in points for moderate n and large p; in particular we have integration formulas of precision $p = 4k - 1$ -- these we could not obtain by the methods of Chapter III. Further, all evaluation points lie within the n-sphere.

CHAPTER V

ERROR BOUNDS FOR n-DIMENSIONAL INTEGRATION FORMULAS OF GAUSSIAN TYPE

To obtain a bound on the error of a repeated Gaussian integration, we proceed as McNamee has done in [19] in the one-dimensional case--in this paper he expresses the error of Gaussian integration as a complex integral and then proceeds to bound this error on a suitable contour in the complex plane. A bound on the value of a complex integrand is often more easily obtained than a bound on the derivative of a moderate or high order of a real integrand.

In the main, the analysis is restricted to integrands which are functions of two variables; corresponding results for n-variable integrands have been derived and are stated in this Chapter without proof.

In section 1) of this Chapter we consider Gauss-Legendre integration. In section 2) we present the analysis of the repeated Gauss-Leguerre case and write down the corresponding result for the Gauss-Hermite case. The results are then extended to repeated integrals that are combinations of Legendre, Laguerre and Hermite integrations.

1) Gauss-Legendre Case

Denoting the zeros of the Legendre polynomial $P_m(x)$, $P_n(y)$ by x_j, y_k respectively, and the corresponding weights by a_j, b_k respectively, we can obtain a bound on the difference

$$(1) \qquad R_2 = \int_{-1}^{1}\int_{-1}^{1} f(x,y)\, dx\, dy - \sum_{j=1}^{n}\sum_{k=1}^{m} a_j b_k\, f(x_j,y_k)$$

by employing contour integration and asymptotic techniques.

Suppose that $f(u,v)$ is an analytic function of two independent complex variables, regular within and on the boundary of regions that enclose the strips $-1 \leqq R\ell\, u \leqq 1$, $-1 \leqq R\ell\, v \leqq 1$ of their respective complex planes. Let C_u be a simple contour that encloses the strip $-1 \leqq R\ell\, u \leqq 1$ and lies within its region of regularity of the complex u-plane. We can then deduce from the contour integral

$$\frac{1}{2\pi i} \int_{C_u} \frac{f(u,y)\, du}{(u - x)P_m(u)}$$

that

(2)
$$f(x,y) = P_m(x) \sum_{j=1}^{m} \frac{f(x_j,y)}{(x - x_j)P'_m(x_j)} + \frac{1}{2\pi i} \int_{C_u} \frac{f(u,y)P_m(x)}{(u - x)P_m(u)}\, du \ .$$

If we integrate this equation with respect to x over $(-1,1)$ and interchange the order of integration in the repeated integral on the right, we obtain

(3)
$$\int_{-1}^{1} f(x,y)\, dx = \sum_{j=1}^{m} \frac{f(x_j,y)}{P'_m(x_j)} \int_{-1}^{1} \frac{P_m(x)}{x-x_j}\, dx$$

$$+ \frac{1}{2\pi i} \int_{C_u} \frac{f(u,y)}{P_m(u)} \int_{-1}^{1} \frac{P_m(x)}{u - x}\, dx\, du$$

We can now employ two known results. If u is not a real number between $(-1,1)$ -- unless it be a zero of $P_m(x)$ -- then

$$2Q_m(u) = \int_{-1}^{1} \frac{P_m(x)}{u - x}\, dx$$

where $Q_m(u)$ is the Legendre function of the second kind. We also have

$$Q_m(u) = \frac{1}{2} P_m(u) \log\left(\frac{u + 1}{u - 1}\right) - f_{m-1}(u)$$

where $f_{m-1}(u)$ is a polynomial of order $m - 1$ (the coefficients of $f_{m-1}(u)$ are tabulated in Whittaker and Watson [20]).

Using these results we can write (3) as

(4)
$$\int_{-1}^{1} f(x,y)\, dx - \sum_{j=1}^{m} a_j f(x_j,y) = \frac{1}{\pi i} \int_{C_u} \frac{f(u,y)\, Q_m(u)}{P_m(u)}\, du \ .$$

The weights a_j are defined by

$$a_j = \int_{-1}^{1} \frac{P_m(x)\,dx}{(x-x_j)P'_m(x_j)} = \frac{2f_{m-1}(x_j)}{P'_m(x_j)} \quad ;$$

they have been extensively tabulated [29, 30, 31].

Setting

$$I = \int_{-1}^{1}\int_{-1}^{1} f(x,y)\,dx\,dy$$

we have

$$I = \int_{-1}^{1} \left\{ \sum_{j=1}^{m} a_j f(x_j,y) + E'_1(y) \right\} dy$$

where $E'_1(y)$ is given on the right side of (4). Repeating the process, we have in view of equation (4)

$$I - \int_{-1}^{1} E'_1(y)\,dy = \sum_{j=1}^{m}\sum_{k=1}^{n} a_j b_k f(x_j,y_k) + E''_2 \quad ,$$

where

(5) $$E''_2 = \frac{1}{\pi i}\int_{C_v} \frac{\sum_{j=1}^{m} a_j f(x_j,v)}{P_n(v)}\, Q_n(v)\,dv \quad ,$$

C_v being an arbitrary contour within the region of regularity of $f(x,v)$, that encloses the strip $-1 \leqq R\ell\, v \leqq 1$ of the complex v-plane. Using equation (4) again we can replace the sum in E''_2 (equation (5)) by its corresponding real and complex integral:

$$\int_{-1}^{1} f(x,v)\,dx - \frac{1}{\pi i}\int_{C_u} \frac{f(u,v)\,Q_m(u)}{P_m(u)}\,du \quad .$$

Recall that C_v is an arbitrary simple contour enclosing the strip $-1 \leqq R\ell\, v \leqq 1$ in the region of regularity of $f(u,v)$. Let any positive number ϵ be given. Then $f(u,v)$ being regular in closed regions

enclosing the strip $(-1,1)$, there exists a $\delta > 0$ and a $\bar{y}$ in $-1 \leqq \bar{y} \leqq 1$ such that $|f(u,v) - f(u,\bar{y})| < \epsilon$ is true for every v on a particular contour for which $0 < |v - \bar{y}| < \delta$. Also, $f(u,v)$ being a regular function of v in regions enclosing the trips $(-1,1)$, the oscillations of $f(u,y)$ with respect to y are finite, and

$$f(u,\bar{y}) = O\left(\int_{-1}^{1} f(u,y)\, dy\right)$$

is true for every $\bar{y}$ in $-1 \leqq \bar{y} \leqq 1$. Thus

$$\frac{1}{\pi i} \int_{C_u} \frac{f(u,\bar{y})\, Q_m(u)}{P_m(u)}\, du = O\left(\int_{C_u} \int_{-1}^{1} \frac{f(u,y)\, Q_m(u)}{P_m(u)}\, dy\, du\right).$$

But the argument on the right hand side of this equation is just

$$\int_{-1}^{1} E_1'(y)\, dy$$

where $E_1'(y)$ is given on the right side of (4) -- this being an error in our Gaussian integration which we assume to be able to estimate (a method of obtaining a bound on this error will be illustrated later) and make appropriately small. Under these assumptions we can write (5) as

$$E_2'' \cong \frac{1}{\pi i} \int_{C_v} \int_{-1}^{1} \frac{f(x,v)\, Q_n(v)}{P_n(v)}\, dx\, dv \doteq E_2$$

since $E_2'' - E_2$ will then only be a negligible second order error.

Combining the results of the above equations, we have

$$\text{(6a)} \qquad \int_{-1}^{1} \int_{-1}^{1} f(x,y)\, dx\, dy - \sum_{j=1}^{m} \sum_{k=1}^{n} a_j b_k f(x_j, y_k) = E_1 + E_2 ,$$

where

$$(6b)\quad \begin{cases} E_1 = \dfrac{1}{\pi i} \displaystyle\int_{C_u} \int_{-1}^{1} \dfrac{f(u,y)\, Q_m(u)}{P_m(u)}\, dy\, du \\[2ex] E_2 = \dfrac{1}{\pi i} \displaystyle\int_{C_v} \int_{-1}^{1} \dfrac{f(x,v)\, Q_n(v)}{P_n(v)}\, dx\, dv \end{cases}$$

Similarly, in n dimensions,

$$(7a)\quad \int_{-1}^{1} \cdots \int_{-1}^{1} f(x^1,\ldots,x^n)\, dx^1 \ldots dx^n - \sum_{j_1=1}^{m_1} \sum_{j_n=1}^{m_n} c^1_{j_1} \ldots c^n_{j_n} \cdot f(x^1_{j_1},\ldots,x^n_{j_n}) = \sum_{i=1}^{n} E_i$$

where

$$(7b)\quad E_j = \frac{1}{\pi i} \int_{C_j} \int_{-1}^{1} \cdots \int_{-1}^{1} \frac{f(x^1,\ldots,u^j,\ldots,x^n)\, Q_{m_j}(u^j)}{P_{m_j}(u^j)}\, dx^1 \ldots dx^{j-1} \cdot dx^{j+1} \ldots dx^n\, du^j$$

where the $x^i_{j_i}$ are zeros of the Legendre polynomial $P_{m_i}(x^i)$ and the $c^i_{j_i}$ are the corresponding weights. Under the assumption that we can appropriately bound each of the E_j in (7b) we have neglected second and higher order errors on the right side of (7a).

The practical use of the error term in (7b) depends on the fact that all contours C_i which enclose the strip $-1 \leqq R\ell\, u^i \leqq 1$ are equivalent if we assume $f(x^1,\ldots,u^i,\ldots,x^n)$ to be regular in a sufficiently large domain, for each x^j, $j \neq i$ in $-1 \leqq x^j \leqq 1$ (We can also take into account a finite number of poles of $f(x^1,\ldots,u^i,\ldots,x^n)$ lying in the complex domain of u^i, and we show this briefly at the end of this section.).

It is convenient to chose as contour a circle of sufficiently large radius R and to employ the asymptotic value, $|u| \to \infty$ of $Q_m(u)/P_m(u)$:

$$(8) \qquad \frac{Q_m(u)}{P_m(u)} = \frac{2^{2m}(m!)^4}{(2m)!(2m+1)!} u^{-2m-1} \{1 + \frac{2m^3+3m^2-m-1}{(2m+3)(2m-1)} u^{-2} + O(u^{-4})\} .$$

As a simple illustration consider

$$f(x,y) = x^4y^2 e^{xy} .$$

In the integration with respect to x we have, according to (7),

$$E_1 = \frac{1}{\pi i} \int_{C_1} \int_{-1}^{1} \frac{u^4y^2 e^{yu} Q_m(u)}{P_m(u)} dy\, du$$

We find on taking only the first term of the asymptotic expansion for $Q_m(u)/P_m(u)$ that the modulus in E_1 is dominated by

$$4y^2K_m \exp\{-[(2m-4)\log R-|y|R]\} \leqq 4K_m \exp\{-[(2m-4)\log R-R]\}$$

on a large circle of radius R, the factorial constant being denoted by K_m. The least value of the function in brackets qua function of R is $R(\log R-1)$; $R = 2m-4$, and a bound on E_1 is given by

$$|E_1| < 4K_m \exp\{-[(2m-4)(\log(2m-4) - 1)]\} .$$

e.g. if

$$m = 6, \qquad |E_1| < 2.56 \times 10^{-7}$$

$$m = 7, \qquad |E_1| < 8.12 \times 10^{-10} .$$

Similarly we bound E_2 where

$$E_2 = \frac{1}{\pi i} \int_{C_2} \int_{-1}^{1} \frac{x^4v^2 e^{xv} Q_n(v)}{P_n(v)} dx\, dv$$

to obtain

$$|E_2| < 4K_n \exp\{-[(2n-2)(\log(2n-2) - 1)]\}$$

e.g. if

$$n = 5, \quad |E_2| < 10.1 \times 10^{-7}$$

$$n = 6, \quad |E_2| < 3.18 \times 10^{-10} .$$

Hence if we set

$$\int_{-1}^{1} \int_{-1}^{1} x^4 y^2 e^{xy} \, dx \, dy - \sum_{i=1}^{m} \sum_{j=1}^{n} a_i b_j \, x_i^4 \, y_j^2 \, e^{x_i y_j} = R_2$$

we have for

$$m = 6, \quad n = 5, \qquad |R_2| < 1.27 \times 10^{-6}$$

$$m = 7, \quad n = 6, \qquad |R_2| < 1.13 \times 10^{-9} .$$

These error bounds could be improved by a more careful analysis but they are simply obtained and not unduly pessimistic, the actual errors being

$$m = 6, \quad n = 5, \qquad |R_2| = 1.49 \times 10^{-8}$$

$$m = 7, \quad n = 6, \qquad |R_2| = 3.36 \times 10^{-11} .$$

If $f(u,y)$ is a meromorphic function, the contribution of the residues at the poles of $f(u,y)$ to the right side of (2) may be supposed to be $g(x,y)$ and (4) is then replaced by

$$(9) \quad \int_{-1}^{1} f(x,y) \, dx - \sum_{i=1}^{m} a_i f(x_i, y) - \int_{-1}^{1} g(x,y) \, dx$$

$$= \frac{1}{\pi i} \int_{C_1} \frac{f(u,y) \, Q_m(u)}{P_m(u)} \, du .$$

The right side of (9) vanishes and the integration formula is exact if $f(u,y)$ is a meromorphic function (of u) and if for every y in $-1 \leqq y \leqq 1$

$$|f(u,y)| \leqq O(|u|^{2m-1}), \quad |u| \to \infty$$

uniformly with respect to the argument of u. For example let $f(u,y)$ be meromorphic, satisfying the above inequality. Let $f(u,y)$ have simple poles (not located on the strip $(-1 \leqq u \leqq 1)$) $\alpha_k(y)$ with residues $A_k(y)$. We then have

$$(10) \qquad \int_{-1}^{1} f(x,y)\,dx = \sum_{i=1}^{m} a_i f(x_i,y) - 2 \sum_{k} \frac{A_k(y)\, Q_m[\alpha_k(y)]}{P_m[\alpha_k(y)]} \quad .$$

From the expansion (8) for $Q_m(u)/P_m(u)$ we see that if the smallest of $|\alpha_k(y)|$ in (10) is sufficiently large, then by increasing m by 1 we would reduce the sum on the right by a factor $\left\{\frac{1}{2|\alpha_k(y)|}\right\}^2$, i.e. we can increase the accuracy in our numerical integration by increasing the number of evaluation points.

If, however, some of the poles $\alpha_k(y)$ are close to the strip $-1 \leqq R\ell\, u \leqq 1$, we may need a large number of points m to reduce the effect of these poles. Although equation (9) indicates an alternative method of evaluating an integral whose integrand has poles in the complex plane, the residue $g(x,y)$ will in most cases be a more complicated function than $f(x,y)$ and it will in general, be better to evaluate the original integral using a larger number of points. This is particularly so in n dimensions since here the poles and residues at the poles may be functions in $n-1$ variables.

By writing $\alpha_k(x^1,\ldots,x^{i-1},x^{i+1},\ldots,x^n)$ for $\alpha_k(y)$ and similarly for A_k, $g(x^1,\ldots,x^i,\ldots,x^n)$ for $g(x,y)$, and making the other appropriate changes corresponding to those between equations (6) and (7), the above discussion of meromorphic functions can easily be extended to n dimensions.

2) Gauss-Laguerre and Gauss-Hermite formulas

Denoting the zeros of the Laguerre polynomials $L_{m_j}(x^j)$ by $x^j_{k_j}$ and the corresponding weights by $c^j_{k_j}$ we illustrate in what follows a method of obtaining a bound on the error

$$(11) \qquad R_n = \int_0^\infty \cdots \int_0^\infty \exp\left(-\sum_{i=1}^n x^i\right) f(x^1,\ldots,x^n)\, dx^1\ldots dx^n - \sum_{j_1=1}^{m_1} \cdots \sum_{j_n=1}^{m_n} c^1_{j_1}\ldots c^n_{j_n}\, f(x^1_{j_1},\ldots,x^n_{j_n})$$

by employing contour integration and asymptotic expansions. The method used in the previous section is applicable here, but some modification is needed since the range of integration is infinite in the Gauss-Laguerre and Gauss-Hermite formulas. We consider the analysis of only the two dimensinal Laguerre formulas in detail, from which we can then write down the corresponding result in n dimensions. The Gauss-Hermite analysis is similar and may be omitted.

Assume that the integral

$$\int_0^\infty\int_0^\infty e^{-(x+y)}\, f(x,y)\, dx\, dy$$

exists and that its value is independent of the order of integration. Assume further that $\exp(-x-y)\, f(x,y) \to \infty$ if either x, or y, or both become infinite, while the other remains zero or positive, and that $f(x,y)$ does

not have any singularities in any finite region of integration. Hence given any $\epsilon > 0$ (Here ϵ may be taken to be a negligible fraction of our allowable error in the computation of the value of the integral.) there exists an M_{11} depending only on ϵ such that

$$(12) \qquad \left| \int_{M_1}^{\infty} e^{-(x+y)} f(x,y)\, dx \right| < \epsilon$$

for each $y \geqq 0$ and for each $M_1 \geqq M_{11}$. A value of M_{22} may similarly be chosen depending only on ϵ, for integration with respect to y.

Let the zeros of the Laguerre polynomial $L_m(u)$ be x_j and let them lie within the strip $(0,M_1)$ of the real u axis. Let C_1 be a contour enclosing the strip $(0,M_1)$ such that u on C_1 satisfies $|u|^{1/k} \leqq M_1 \leqq |u| - \delta$, where $0 < \delta \leqq \frac{1}{4} M_1$ and $k > 1$ are otherwise arbitrary positive numbers. Then if $f(u,v)$ is a regular function of u in a region enclosing this contour for every complex number v in a similar region of regularity, we can write,

$$f(x,y) = \sum_{i=1}^{m} \frac{f(x_i,y)\, L_m(x)}{(x-x_i)\, L_m'(x_i)} + \frac{1}{2\pi i} \int_{C_1} \frac{f(u,y)\, L_m(x)}{(u-x)\, L_m(u)}\, du$$

where y is in $0 \leqq y \leqq M_2$, $M_2 \geqq M_{22}$. On multiplying this equation by e^{-x}, integrating with respect to x over $(0,M_1)$ and interchanging the order of integration in the double integral we obtain

$$(13) \qquad \int_0^{M_1} e^{-x} f(x,y)\, dx - \sum_{i=1}^{m} \frac{f(x_i,y)}{L_m'(x_i)} \int_0^{M_1} \frac{e^{-x}\, L_m(x)}{x - x_i}\, dx$$

$$= \frac{1}{2\pi i} \int_{C_1} f(u,y) \int_0^{M_1} \frac{L_m(x)\, e^{-x}\, dx}{L_m(u)(u-x)}\, du .$$

We now proceed to find the asymptotic expansion of the inner integral on the right of this equation. Our asymptotic sequence will be $\{\frac{1}{u}\}$, $|u| \to \infty$; hence any term which is $O[|u|^r \exp(-|u|^{1/k})]$ where r and k $(k > 1)$ are arbitrary positive numbers, will not contribute towards our asymptotic expansion and is "asymptotically equal to zero" with respect to our asymptotic sequence. This can be written more concisely as $O(|u|^r \exp(-|u|^{1/k})) \approx 0; \ \{\frac{1}{u}\}$.

Our first step is to show that

$$(14) \qquad \int_{M_1}^{\infty} \frac{e^{-x} L_m(x)}{u - x} dx \approx 0 \ ; \quad \{\tfrac{1}{u}\} \quad .$$

If the point u is not on the axis of integration, this can easily be shown. For suppose $u = \sigma + i\omega$, where σ and ω are real, $|\omega| \geqq \alpha/|u|^r$, α and r being arbitrary positive numbers. Then

$$\left| \int_{M_1}^{\infty} \frac{e^{-x} L_m(x)}{u - x} dx \right| \leqq \frac{|u|^r}{\alpha} \left| \int_{M_1}^{\infty} e^{-x} L_m(x)\, dx \right| \quad .$$

Now for $k > 0$ but otherwise an arbitrary positive number, it is readily shown that

$$\left| \int_{M_1}^{\infty} e^{-x} x^k \, dx \right| < 2(2M_1)^k \exp(-M_1) \ ,$$

and therefore

$$\left| \int_{M_1}^{\infty} \frac{e^{-x} L_m(x)}{u - x} dx \right| < O[(2M_1)^m |u|^r \exp(-M_1)]$$

$$\leqq O[\exp(-|u|^{1/k}) |u|^{r+m} \cdot 2^m] \approx 0 \ ; \quad \{\tfrac{1}{u}\} \quad .$$

If, on the other hand u is on the positive x-axis, we evaluate the Cauchy principal value of the integral:

$$P\int_{M_1}^{\infty} \frac{e^{-x} L_m(x)}{u - x}\, dx = \int_{M_1}^{u-\omega} + P\int_{u-\omega}^{u+\omega} + \int_{u+\omega}^{\infty}$$

where ω is some number in $\frac{\alpha}{|u|^r} \leqq \omega \leqq \delta$, $0 < \delta \leqq \frac{1}{4} M_1$, α and r being arbitrary positive numbers. From the above analysis we see that the first and last integrals on the right are asymptotically equal to zero with respect to our asymptotic sequence, $\{\frac{1}{u}\}$.

To show that the center integral is asymptotically equal to zero with respect to our asymptotic sequence, we let $x = u + t$ when $x \geqq u$, and $x = u - t$ when $x \leqq u$ in the integral to obtain

$$P\int_{u-\omega}^{u+\omega} \frac{e^{-x} L_m(x)}{u - x}\, dx = \int_0^{\omega} \frac{e^{-(u+t)} L_m(u+t) - e^{-(u-t)} L_m(u-t)}{-t}\, dt.$$

The integral on the right obviously exists since the integrand is bounded everywhere--at $t = 0$ it has the value $-2\frac{d}{dx}\{e^{-x} L_m(x)\}\Big|_{x=u}$. By the mean value theorem of the differential calculus, given any t in $0 \leqq t \leqq \omega$ there is a point λ in $-t < \lambda < t$ such that

$$\frac{e^{-u-t} L_m(u+t) - e^{-u+t} L_m(u-t)}{-t} = -2\frac{d}{dx}\{e^{-x} L_m(x)\}\Big|_{x=u+\lambda}$$

Let the maximum value of this expression in $-\omega \leqq \lambda' \leqq \omega$ be at the point $x' = u + \lambda'$. Hence on replacing the above integral on the right by the maximum value of the integrand in the interval times ω, we find that (15) is satisfied even if the point u is on the positive real axis. Therefore

$$\int_0^{M_1} \frac{e^{-x} L_m(x)}{u - x}\, dx \approx \int_0^{\infty} \frac{e^{-x} L_m(x)}{u - x}\, dx \;;\; \{\frac{1}{u}\}$$

whenever $|u|^{1/k} \leqq M_1 \leqq |u| - \delta$, $k > 1$, $0 < \delta \leqq \frac{1}{4} M_1$, $|u| \to \infty$, the integral on the right being assumed to be a Cauchy principal value when the point u lies on the positive real axis.

Expanding the denominator in the above integrand in powers of x/u we note that

$$\int_0^\infty \frac{e^{-x} L_m(x)}{u - x} dx = \int_0^\infty \frac{e^{-x} (x/u)^m L_m(x)}{u - x} dx ,$$

the first m terms of our expansion integrating to zero due to the orthogonality property of the Laguerre polynomials. Thus under the conditions stated above

$$\int_0^{M_1} \frac{e^{-x} L_m(x)}{u - x} dx = \sum_{j=0}^{N} \frac{\beta_j^1}{u^{j+m+1}} + O(u^{-N-m-2})$$

where

$$\beta_j^1 = \int_0^\infty e^{-x} x^{m+j} L_m(x) dx .$$

The inverse of the Laguerre polynomial $L_m(u)$ may also be expanded in powers of $1/u$, $|u| \to \infty$ to give

$$\frac{1}{L_m(u)} = (-1)^m u^{-m} m! \; [1 + m^2 u^{-1} + O(u^{-2})]$$

and we finally have

$$(15) \qquad \int_0^{M_1} \frac{e^{-x} L_m(x) \, dx}{(u-x) \, L_m(u)} = \sum_{j=0}^{N} \frac{a_j^1}{u^{2m+j+1}} + O(u^{-2m-N-2}) ,$$

$|u| \to \infty$; the first two a^1's being given by

$$a_0^1 = (m!)^2, \quad a_1^1 = (2m^2 + 2m + 1)(m!)^2 .$$

By equation (12) and the above arguments we can also neglect the contribution of the integration from M_1 to ∞ of the left hand side of (13) to our asymptotic expansion. Thus, with

$$c_k^1 = \int_0^\infty \frac{e^{-x} L_m(x)\, dx}{L_m'(x_k)(x-x_k)}$$

we obtain

$$(16) \qquad \int_0^\infty e^{-x} f(x,y)\, dx - \sum_{k=1}^{m} c_k^1 f(x_k,y)$$

$$\sim \frac{1}{2\pi i} \int_{C_1} f(u,y) \sum_{j=0}^{\infty} a_j^1 u^{-2m-j-1}\, du,$$

$|u| \to \infty$ $\quad 0 \leqq y \leqq M_2$, M_2 being defined similarly for y as M_1 was for x, by equation (12).

We can thus write

$$(17a) \qquad \int_0^\infty \int_0^\infty e^{-x-y} f(x,y)\, dx\, dy - \sum_{i=1}^{m} \sum_{j=1}^{n} c_i^1 c_j^2 f(x_k,y_j) = E_1 + E_2$$

where

$$(17b) \qquad \begin{cases} E_1 \sim \dfrac{1}{2\pi i} \displaystyle\int_{C_1} \int_0^\infty e^{-y} f(u,y)\, dy\, \{(m!)^2 (u)^{-2m-1} + \ldots\}\, du, \quad |u| \to \infty \\[2ex] E_2 \sim \dfrac{1}{2\pi i} \displaystyle\int_{C_2} \int_0^\infty e^{-x} f(x,v)\, dx\, \{(n!)^2 (v)^{-2n-1} + \ldots\}\, dv \quad |v| \to \infty \end{cases}$$

and where it can be shown by an analysis similar to that given for the Gauss-Legendre case that if the bound on the integrals given by equation (17b) can be made suitably small then the term neglected on the right side of (17a) is a second order negligible error (i.e. the error is of order E_1E_2.).

Our results have the following obvious extension to n dimensions,

$$\text{(18a)}\qquad \int_0^\infty \cdots \int_0^\infty \exp\Big(-\sum_{i=1}^{n} x^i\Big) f(x^1,\ldots,x^n)\, dx^1 \ldots dx^n$$

$$-\sum_{k_1=1}^{M_1} \cdots \sum_{k_n=1}^{M_n} c^1_{k_j} \cdots c^n_{k_j}\, f(x^1_{k_1},\ldots,x^n_{k_n}) = \sum_{i=1}^{n} E_i$$

where

$$\text{(18b)}\qquad E_j \sim \frac{1}{2\pi i} \int_{C_i} \int_0^\infty \int_0^\infty \exp\Big(x^j - \sum_{i=1}^{n} x^i\Big) f(x^1,\ldots,x^{j-1}, u^j, x^{j+1}, \ldots, x^n)\ .$$

$$.\ dx^1 \ldots dx^{j-1} dx^{j+1} \ldots dx^n \left\{ \frac{(m_j!)^2}{(u^j)^{2m_j+1}} + \ldots \right\} du^j$$

$|u^j| \to \infty$, and where second and higher order error terms have been neglected on the right side of (18a), assuming that we can suitably bound each of the E_i's.

Under conditions similar to those given for Gauss-Laguerre integration we also write formulas for errors involved when performing Gauss-Hermite integration. Here we have

$$\text{(19a)}\qquad \int_{-\infty}^\infty \cdots \int_{-\infty}^\infty \exp\Big(-\sum_{i=1}^{n} (x^i)^2\Big) f(x^1,\ldots,x^n)\, dx^1 \ldots dx^n$$

$$-\sum_{k_1=1}^{m_1} \cdots \sum_{k_n=1}^{m_n} c^1_{k_j} \cdots c^n_{k_j}\, f(x^1_{k_1},\ldots,x^n_{k_n}) = \sum_{i=1}^{n} E_i$$

where in this case $x^i_{k_i}$ are zeros of the Hermite polynomial $H_{m_i}(x^i)$ and the $c^i_{k_i}$ are the corresponding weights. The errors E_i on the right side of (19a) are given by

$$(19b) \qquad E_j \sim \frac{1}{2\pi i} \int_{C_j} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \exp[(x^j)^2 - \sum_{i=1}^{n} (x^i)^2]\, f(x^1,\ldots,x^{j-1}, u^j, x^{j+1},\ldots,x^n)\cdot$$

$$\cdot dx^1 \ldots dx^{j-1} dx^{j+1} \ldots dx^n \left\{ \frac{\pi m_j!}{2^m} (u^j)^{-2m_j-1} \left[1 + \frac{(m_j)^2 + 2m_j + 1}{4(u^j)^2} + O((u^j)^{-4}) \right] \right\} \cdot du^j$$

$|u^j| \to \infty$. Assuming we can suitably bound each E_i on the right side of (19b) we have neglected second and higher order errors on the right side of (19a).

We now return again to our two-dimensional model for simplicity. If $f(u,y)$ is a meromorphic function with poles in the complex u-plane, the contribution of the residues at the poles of $f(u,y)$ to our contour integration may be supposed to be $g(x,y)$, and (16) is then replaced by

$$(20) \qquad \int_0^\infty e^{-x} f(x,y)\, dx - \sum_{i=1}^{m} c_i^1 f(x_i,y) - \int_0^\infty e^{-x} g(x,y)\, dx$$

$$\sim \frac{1}{2\pi i} \int_{C_1} f(u,y) \left\{ \frac{(m!)^2}{u^{2m+1}} + \ldots \right\} du, \quad |u| \to \infty ,$$

while $0 \leqq y \leqq M_2$.

The right side of (20) vanishes and the integration formula is exact if $f(u,y)$ is a meromorphic function of u and if for every y in $0 \leqq y \leqq M_2$

$$|f(u,y)| \leqq O(|u|^{2m-1}), \quad |u| \to \infty$$

uniformly with respect to the argument of u. For example, let $f(u,y)$ be

meromorphic, satisfying the above inequality. Let $f(u,y)$ have simple poles (not located on the strip $0 \leqq u < \infty$) $\alpha_k(y)$ with residues $A_k(y)$. We then have

$$(21) \qquad \int_0^\infty e^{-x} f(x,y)\,dx - \sum_{i=1}^{m} c_i^1 f(x_i,y) \sim \sum_k A_k(y) \left\{ \frac{(m!)^2}{[\alpha_k(y)]^{2m+1}} + \cdots \right\}$$

If the $\alpha_k(y)$ are sufficiently far from the origin, the contribution of the residues to the integrand will be negligible. Since, as $m \to \infty$, the right hand side of (21) becomes unbounded*, we may want to use formula (20) to evaluate the original integral to the accuracy desired. This, however, may become a very complicated problem, since the function $g(x,y)$ will in general be more complicated than the function $f(x,y)$.

The above discussion applies equally to the Hermite formulas. It can also easily be extended to the n-dimensional case. In the n-dimensional case the poles and residues at the poles may be functions in $(n-1)$ variables, as we have already noted in our treatment of the Gauss-Legendre formulas.

In closing, we note that by our procedure we can obtain error bounds for multiple integrals over combinations of regions, e.g. for integrals of the form

$$\left(\int_{-1}^{1}\right)^{i}\left(\int_{0}^{\infty}\right)^{j}\left(\int_{-\infty}^{\infty}\right)^{k} \exp[-x^{i+1}-\ldots-x^{i+j}-(x^{i+j+1})^2-\ldots-(x^{i+j+k})^2]\cdot$$
$$\cdot f(x^1,\ldots,x^{i+j+k})\,dx^1\ldots dx^{i+j+k} \quad .$$

As an example, consider evaluating

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* Note, however, there is a "best" m.

$$\int_{z=-\infty}^{\infty}\int_{y=0}^{\infty}\int_{x=-1}^{1} \frac{100e^{-y-z^2} J_0(\tfrac{1}{2}y)\cos z}{x^2 - 4x + 104}\, dx\, dy\, dz = \frac{20}{\sqrt{5}}(\tan^{-1}.3-\tan^{-1}.1)\cdot \pi\, e^{-.25}$$

by a repeated Gaussian integration formula*.

By our previously developed formulas,

$$E_1 \sim \frac{1}{\pi i}\int_{C_1}\int_0^{\infty}\int_{-\infty}^{\infty} \frac{100e^{-y-z^2} J_0(\tfrac{1}{2}y)\cos z}{u^2 - 4u + 104}\, dz\, dy \left\{\frac{2^{2j}(j!)^4 u^{-2j-1}}{(2j)!(2j+1)!} + \ldots\right\} du$$

$$|u|\to\infty ,$$

$$E_2 \sim \frac{1}{2\pi i}\int_{C_2}\int_{-\infty}^{\infty}\int_{-1}^{1} \frac{100e^{-z^2} J_0(\tfrac{1}{2}v)\cos z}{x^2 - 4x + 104}\, dx\, dz \left\{\frac{(m!)^2}{v^{2m+1}} + \ldots\right\} dv$$

$$|v|\to\infty ,$$

$$E_3 \sim \frac{1}{2\pi i}\int_{C_3}\int_0^{\infty}\int_{-1}^{1} \frac{100e^{-y} J_0(\tfrac{1}{2}y)\cos w}{x^2 - 4x + 104}\, dx\, dy \left\{\frac{\pi\, n!}{2^n w^{2n+1}} + \ldots\right\} dw$$

$$|w|\to\infty .$$

By taking u sufficiently large we observe that $E_1 = 0$. The denominator of the integrand as a function of u has poles at the points α_1 and α_2 ($\alpha_1 = 2 + 10i$, $\alpha_2 = 2 - 10i$), where $|\alpha_s| > 10$. Both $J_0(\tfrac{1}{2}y)$ and $\cos z$ are less than or equal to one in magnitude. Hence expanding the sum in the residues of the right hand side of (7) asymptotically, i.e. using the

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* Note: This integral can be evaluated with $j + m + n$ points instead of with jmn, by writing it in the form $\int_{-\infty}^{\infty} F(z)dz \int_0^{\infty} G(y)dy \int_{-1}^{1} H(x)dx.$

asymptotic expansion for $Q_j(\alpha_s)/P_j(\alpha_s)$, we find that the dominating term of the error E_1 is less than

$$100\pi \cdot \frac{2^{2j}\,(j!)^4}{(2j)!(2j+1)!} \cdot \frac{2}{|\alpha_s|^{2j+1}} \cong 3.14 \times 10^{-5} \quad \text{for} \quad j = 2.$$

For x in $-1 \leqq x \leqq 1$, $100/(x^2 - 4x + 104) < 1$. $J_0(\frac{1}{2}v)$ has a maximum value for v imaginary. Along an imaginary axis

$$|J_0(\tfrac{1}{2}v)| = \tfrac{1}{2}\left(\frac{1}{\pi|v|}\right)^{\frac{1}{2}} \cosh \frac{|v|}{2} \left(1 + O\left(\frac{1}{|v|}\right)\right).$$

Hence considering only the first term of the asymptotic expansion indicated in E_2 above together with the dominating term of the asymptotic expansion for $|J_0(\frac{1}{2}v)|$, v imaginary, we find that the dominating term in E_2 has a minimum value on a large circle $|v| = 4m + 1$. Here

$$|E_2| \leqq 2\pi(4m + 1)^{1/2}\, e^{-3/2} \left(\frac{m + 1}{4m + 1}\right)^{2m+1}$$

$$\cong 9.45 \times 10^{-5} \quad \text{with} \quad m = 4 .$$

Similarly

$$|E_3| \leqq \frac{2\pi}{e}\,(n + 1)\cdot\left(\frac{(n + 1)e}{8n^2}\right)^n \cong 1.56 \times 10^{-5} \quad \text{with} \quad n = 5.$$

The overall estimate of the error in the triple sum is $|E_1| + |E_2| + |E_3| \leqq 1.4 \times 10^{-4}$. The actual error when evaluating the integral by a numerical integration formula is 7.8×10^{-6}, indicating that our error bounds are not too pessimistic; we could have improved them by estimating our integrals more carefully.

Notice that in our last example we have used our formulas for error bounds to find the required number of points for achieving a desired accuracy of numerical integration.

CHAPTER VI

SUMMARY, AND SOME UNSOLVED PROBLEMS

This Chapter states unsolved problems and summarizes the results presented in the thesis.

If we survey in retrospect the development of numerical integration formulas we discern a rapid advancement in the one-dimensional cases during the time of Gauss and Chebychev. With the help of orthogonal polynomials the results of these authors have been extended to many different ranges of integration for many different weight functions. Indeed, every set of orthogonal polynomials provides a new integration formula.

For a number of reasons, numerical integration formulas in higher dimensions have developed more slowly. Before the development of electronic digital computers (around 1947) no practical use could be found for an integration formula in higher dimensions. The availability of powerful computing devices has however made possible the study of functions of several independent variables, and new attempts at obtaining numerical integration formulas in higher dimensions are being made.

Among the formulas available in higher dimensions are those over rectangular regions that can be obtained from one-dimensional integration formulas by taking products of integrals over each separate variable. As the number of dimensions increases the number of evaluation points required by these formulas increases so rapidly that the economic limit on even the most powerful digital computers is very quickly reached.

An estimation of the minimum number of points required by an integration formula of precision p tells us that the integration formulas resulting from taking products of integrals over each separate variable are

extravagant in points when the number of dimensions n is large; that is, integration formulas exist which require less than $(\frac{p+1}{2})^n$ points to integrate an arbitrary monomial of degree $\leqq p$ over an arbitrary region in n-space.

This class of "minimum-point" formulas is, however not easily obtained. We have seen earlier that the main difficulty is the solution of systems of non-linear algebraic equations. In the one-dimensional case there exists a beautiful theory of orthogonal polynomials which enables one to obtain numerical integration formulas with little difficulty. This theory is readily extended to rectangular regions in n-space, where it enables one to obtain $(\frac{p+1}{2})^n$ - point formulas. Unfortunately no such theory exists as yet for the class of minimum-point formulas. The development of such a theory, or else the development of effective methods of solving a system of non-linear algebraic equations would solve the problem of obtaining minimum-point numerical integration formulas of precision p over regions in n-space.

A great deal of work on obtaining numerical integration formulas in higher dimensions has been done by P. C. Hammer and A. H. Stroud. They have given numerical integration formulas up to precision 3 for various regions in n-space, and formulas up to precision 5 for the n-cube and the n-sphere. In cases where a transformation is available between two regions they have given a method for extending numerical integration formulas from one region to the other region. They have introduced and applied the concept of fully symmetric regions which reduces considerably the number of simultaneous non-linear algebraic equations we need to solve to obtain numerical integration formulas over regions in n-space.

Numerical integration formulas of precision 3, 5 and 9 for arbitrary fully symmetric regions in n-space are developed in this thesis. The

Apart from conceivable fully symmetric regions and weight functions for which equations such as (10) on page 44 do not hold the method by which these formulas were obtained will lead to fully symmetric numerical integration formulas of precision $4k + 1$. The non-linear unknowns in the simultaneous non-linear algebraic equations set up to obtain numerical integration formulas are shown to be the zeros of polynomials in one variable orthogonal over the fully symmetric region with respect to the fully symmetric weight function.

To overcome the restriction the above equation imposes on p ($p \neq 4k - 1$) we turn to the repeated Gaussian formulas, noting that for moderate n and large p these formulas are not unduly extravagant in evaluation points inasmuch as they produce greater accuracy for large p. A transformation is given, by which we can transform the integral

$$\int_a^b \int_{\varphi^2(x^1)}^{\psi^2(x^1)} \cdots \int_{\varphi^2(x^1,\ldots,x^{n-1})}^{\psi^n(x^1,\ldots,x^{n-1})} f(x^1,\ldots,x^n)dx^n \ldots dx^1$$

into an integral over the n-cube. This transformation serves to broaden the range of application of the Gaussian formulas, and suggests that instead of looking for numerical integration formulas over arbitrary (bounded) regions in n-space we may restrict ourselves to the simpler problem of looking for numerical integration formulas for the n-cube, varying only the weight function. For completeness, a $2(\frac{p+1}{2})^n$-point numerical integration formula of arbitrary high precision p is developed for the finite and infinite n-sphere, where $n \geqq 2$.

Error bounds are necessary. The majority of the a priori estimates require an estimate in the form of a $(p+1)$-th derivative of the integrand while in most cases empirical estimates are more easily obtained than moderate or high order derivative estimates. In this thesis we have utilized con-

tour integration and asymptotics to obtain error bounds for repeated Gaussian integration in the case where the integrand is a meromorphic function of n complex variables.

The Appendix of the thesis consits of (A) a number of examples of numerical integrations over regions in 4-space and (B) Table V which is a tabulation of the zeros and weights of four particular fully symmetric numerical integration formulas of precision 9.

Computors may ask what positive recommendation can be made. The field has not yet been explored to the extent that precise prescriptions are possible, but it may be useful to give some broad indications.

In the one-dimensional case it is easy to find examples of integrands favorable to one type of numerical integration formula and unfavorable to another. This suggests that since we are unlikely to find omnibus integration formulas in the one dimensional case, we are even less likely to find such formulas in n dimensions. In practice we use Gaussian integration in the one dimensional case, tolerating a certain amount of extravagance if necessary. Extravagance is unfortunately less tolerable in n-space, and in order to keep the number of points at a feasible level the choice of the integration formula will depend to a large extent on the integration problem.

Higher dimensional integrals often come up in computing centers--mainly from nuclear physics and chemical engineering. No attempt has been made to collect examples of these or to classify them, although this would be desirable.

At present, moderate precision in higher dimensional integration is relatively easy to obtain. High precision is also relatively easy to

obtain, but may sometimes be costly. This depends largely on the complexity of the integrand, the number of dimensions, and the precision desired. The error of numerical integration is sometimes tractable, though usually it is intractable by reason of the complexity of the integrand. Usually the numerical integration is repeated with a formula of higher precision and the results of the two integrations are compared. This procedure is somewhat unsatisfactory since it can be misleading.

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APPENDIX A

EXAMPLES OF NUMERICAL INTEGRATIONS

In this Appendix we give examples to illustrate the accuracy of some of the numerical integration formulas developed in this thesis. The integrals were evaluated on the I.B.M. 1620 at the University of Alberta.

1) $$I_\alpha = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} [w + x + y + z + 5]^\alpha \, dw\, dx\, dy\, dz$$

The results of this example are given in Table VII. This Table contains:

(i) the value of α

(ii) I_α, the exact value of the integral

(iii) relative errors $E(p3)$, $E(p5)$, $E(p9A)$, $E(p9B)$. By relative error is meant the absolute value of the actual error, $S_m - I_\alpha$ divided by I_α, where S_m is obtained by numerical integration.

The notation $E(p3)\ldots E(p9B)$ is explained below.

$E(p3)$ is the relative error of the 8-point formula of precision 3 for the 4-cube given on pages 41-43.

$E(p5)$ is the relative error of the 33-point formula of precision 5 for the 4-cube given on pages 41-43.

$E(p9A)$ and $E(p9B)$ are relative errors of the 177-point formulas of precision 9 given on pages 42-46 and tabulated on page 109 of Table V (Appendix B). Formula A is the formula tabulated on the left hand side of this table; formula B is tabulated on the right hand side.

TABLE VII [EXAMPLE 1)]

α	I_α	E(p3)	E(p5)	E(p9A)	E(p9B)
-.5	7.3166065	$1.6x10^{-3}$	$2.2x10^{-4}$	$7.9x10^{-5}$	$1.0x10^{-4}$
.5	35.525901	$2.4x10^{-4}$	$3.7x10^{-5}$	$1.2x10^{-5}$	$1.1x10^{-5}$
1.5	182.49692	$1.2x10^{-4}$	$3.3x10^{-6}$	$4.1x10^{-6}$	$5.1x10^{-6}$
2.0	421.33189	$3.6x10^{-6}$ *	$3.0x10^{-6}$ *	$7.0x10^{-6}$ *	$2.3x10^{-6}$ *
2.5	983.59047	$1.8x10^{-4}$	$6.9x10^{-6}$	$3.0x10^{-6}$	$2.6x10^{-6}$
4.0	13,276.768	$3.6x10^{-3}$	$1.9x10^{-6}$ *	$2.3x10^{-6}$ *	$2.1x10^{-6}$ *
6.0	479,213.79	$3.9x10^{-1}$	$4.4x10^{-4}$	$2.9x10^{-7}$ *	$4.0x10^{-6}$ *
8.0	19,236,265	$1.2x10^{-1}$	$8.0x10^{-3}$	$2.1x10^{-7}$ *	$1.0x10^{-6}$ *
10.0	840,845,570	$2.5x10^{-1}$	$3.7x10^{-2}$	$3.2x10^{-6}$	$1.0x10^{-5}$

2) $$I_\alpha = \underset{4\text{-sphere}}{\iiiint} \frac{dw\ dx\ dy\ dz}{(1 + w^2 + x^2 + y^2 + z^2)^\alpha}$$

The results of this example are given in Table VIII. α, I_α, E(p3), E(p5), E(p9A) and E(p9B) are defined as in the previous example. The precision 3 and 5 formulas are located on pages 41-43. The precision 9 formulas are given on pages 42-46 and tabulated on page 119 of Table V (Appendix B).

ER1[+], ER2[+] and ER3[+] are relative errors of the $2(m)^n$-point formulas of precision 2m-1 over the n-sphere derived in section 3) of Chapter IV.

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* Theoretically, the relative errors should have vanished in these cases. The errors given are due to an accumulation of round-off errors; 8 significant figures were carried in the calculations.

TABLE VIII

RELATIVE ERRORS OF INTEGRATIONS OVER THE 4-SPHERE [EXAMPLE 2)]

α	I_α	E(p3)	E(p5)	E(p9A)	E(p9B)	ER1$^+$	ER2$^+$	ER3$^+$
-3.5	31.979939	7.7×10^{-2}	1.0×10^{-2}	1.4×10^{-6}	6.5×10^{-6}	1.3×10^{-1}	4.3×10^{-3}	4.0×10^{-5}
-3.0	24.180560	5.5×10^{-2}	5.1×10^{-3}	1.1×10^{-6} *	1.5×10^{-6} *			
-2.5	18.350760	3.5×10^{-2}	1.7×10^{-3}	1.6×10^{-7}	2.7×10^{-6}	7.4×10^{-2}	1.1×10^{-3}	1.5×10^{-6}
-1.5	10.698976	7.6×10^{-3}	3.9×10^{-4}	2.3×10^{-6}	3.0×10^{-6}	3.1×10^{-2}	7.7×10^{-5}	5.6×10^{-7}
-1.0	8.2246737	7.2×10^{-7} *	4.5×10^{-7} *	4.0×10^{-7} *	4.9×10^{-7} *	1.6×10^{-2}	6.3×10^{-7}*	4.9×10^{-8} *
-.5	6.3539571	2.6×10^{-3}	4.5×10^{-4}	5.3×10^{-6}	3.0×10^{-6}	5.4×10^{-3}	2.2×10^{-5}	1.1×10^{-6}
1.5	2.3947634	4.2×10^{-2}	2.0×10^{-2}	7.7×10^{-4}	7.7×10^{-3}	1.9×10^{-2}	1.4×10^{-3}	3.8×10^{-5}
2.5	1.5280378	1.0×10^{-1}	6.7×10^{-2}	3.7×10^{-3}	3.4×10^{-2}	6.2×10^{-2}	4.0×10^{-3}	4.8×10^{-5}

* See the footnote on page 100.

$ER1^+$ is the relative error of the 2-point formula of precision 1;

$ER2^+$ is the relative error of the 32-point formula of precision 3;

$ER3^+$ is the relative error of the 162-point formula of precision 5.

The two examples given so far serve to indicate that branch points in the integrand have a less destructive effect on the accuracy of numerical integration than poles in the complex plane.

Note also that formula A of precision 9 tends to give slightly better results than formula B. The most likely reason for this is that the weights in formula A have a better distribution of magnitudes than those in formula B. This suggests that we could achieve better accuracy in our numerical integration by first carrying out the sums that are multiplied by the smallest weights. The accuracy of the $2(m)^n$ -point formulas are surprising. The extremely large number of points required would however, make them useless when the number of dimensions is large; this rapid growth of required evaluation points does not occur for the other formulas.

3) $$I_\alpha = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \frac{[w+x+y+z+5]^\alpha dw\, dx\, dy\, dz}{[(1-w^2)(1-x^2)(1-y^2)(1-z^2)]^{\frac{1}{2}}}$$

The results of this example are given in Table IX. In this Table, α, I_α, E(p3), E(p5), E(p9A) and E(p9B) are defined as in Example 1). The precision 3 and 5 formulas are located on pages 41-43. The precision 9 formulas are given on pages 42-46 and tabulated on page 129 of Table V (Appendix B). Formula A is the formula tabulated on the left hand side of this Table; formula B is tabulated on the right hand side.

TABLE IX [EXAMPLE 3)]

α	I_α	E(p3)	E(p5)	E(p9A)	E(p9B)
-.5	45.159877	4.2×10^{-3}	1.2×10^{-4}	2.1×10^{-5}	5.7×10^{-5}
.5	215.45965	5.2×10^{-4}	1.0×10^{-4}	6.7×10^{-6}	1.4×10^{-6}
1.5	1,122.2089	2.7×10^{-4}	2.2×10^{-5}	4.5×10^{-6}	4.1×10^{-6}
2.0	2,630.0118	1.3×10^{-5} *	1.1×10^{-5} *	1.3×10^{-5} *	1.1×10^{-5} *
2.5	6,258.3709	3.8×10^{-4}	1.8×10^{-5}	5.6×10^{-6}	4.6×10^{-6}
4.0	91,125.664	6.9×10^{-3}	3.7×10^{-6} *	4.6×10^{-6} *	4.6×10^{-6} *
6.0	3,739,737.7	1.7×10^{-2}	1.2×10^{-3}	1.1×10^{-5} *	1.2×10^{-5} *
8.0	173,555,240	1.9×10^{-1}	1.8×10^{-2}	1.4×10^{-5} *	1.2×10^{-5} *
10.0	8,836,719,600	3.5×10^{-1}	7.3×10^{-2}	1.0×10^{-5}	2.4×10^{-5}

4) $$I_\alpha = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\exp(-w^2-x^2-y^2-z^2)\ dw\ dx\ dy\ dz}{[1+w^2+x^2+y^2+z^2]^\alpha}$$

The results of this example are given in Table X. All notations are the same as those for Example 1). The integration formulas of precision 9 are tabulated on page 139 of Table V (Appendix B).

The results of Example 3) (Table IX) are better than the results of Example 4 (Table X). This is largely due to the integrand being similar in complexity in the two examples, while the region of integration is larger in Example 4). We note again, that the destructive effect on the accuracy of the numerical integration is much more due to poles in the complex regions of the integrand than to branch points. Also, formula A of precision 9

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* See footnote page 100.

gives somewhat better results than formula B; this being largely due to the weights in formula A having a much better distribution of magnitudes.

TABLE X [EXAMPLE 4)]

α	I_α	E(p3)	E(p5)	E(p9A)	E(p9B)
-3.4	1,092.7610	5.7×10^{-1}	2.3×10^{-1}	4.4×10^{-4}	2.4×10^{-3}
-3.0	483,61046	4.5×10^{-1}	1.2×10^{-1}	4.1×10^{-8} *	4.8×10^{-7} *
-2.5	223.81810	3.1×10^{-1}	4.5×10^{-2}	1.8×10^{-4}	1.1×10^{-3}
-1.5	55.295497	7.3×10^{-2}	1.1×10^{-2}	1.5×10^{-4}	1.2×10^{-3}
-1.0	29.608802	1.0×10^{-7} *	3.4×10^{-7} *	2.0×10^{-7} *	2.4×10^{-7} *
- .5	16.674363	2.5×10^{-2}	1.3×10^{-2}	2.4×10^{-4}	2.7×10^{-3}
1.5	2.7013433	3.0×10^{-1}	5.2×10^{-1}	1.1×10^{-3}	1.9×10^{-1}
2.5	1.3886772	5.4×10^{-1}	1.5	2.0×10^{-2}	6.3×10^{-1}

5) $$I = \iiiint\limits_{\text{4-sphere}} \exp(-w^2-x^2-y^2-z^2)\, dw\, dx\, dy\, dz$$

This example is due to J. H. Cadwell [23]; in this paper an Algol program is given to evaluate an integral of the type (2), Chapter IV (page 64) using Simpson's rule** with repeated bisection to attain the required accuracy. In this paper the author has, in effect, utilized the transformation given in Chapter IV of this thesis. We present the results of this example in Table XI.

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* See the footnote on page 100.

** Simpson's Rule: $\int_{x_0}^{x_0+2kh} f(x)dx = \frac{1}{3}[f_0+4f_1+2f_2+4f_3+\ldots+2f_{2k-2}+$ $+4f_{2k-1}+f_{2k}]$ where $f_j = f(x_0+jh)$.

E(p3), E(p5), E(p9A), E(p9B), ER1, ER2 and ER3 have the same meaning as in Example 2). ER(S1), ER(S2) and ER(S3) are the relative errors by the method in [23]. The number of evaluation points is also given.

TABLE XI [EXAMPLE 5)]

Exact Value 2.6079550	Relative Errors	No. of Evaluation points
E(p3)	2.8×10^{-2}	8
E(p5)	4.8×10^{-3}	33
E(p9A)	4.6×10^{-6}	177
E(p9B)	2.5×10^{-5}	177
ER1	2.2×10^{-3}	2
ER2	3.5×10^{-4}	32
ER3	1.0×10^{-5}	162
ER(S1)	6.2×10^{-2}	625
ER(S2)	6.2×10^{-3}	3,345
ER(S3)	6.2×10^{-4}	69,113

Note again, the surprising accuracy of the $2(m)^n$ - point formulas constructed over the n-sphere, and that the 177-point formula A of precision 9 gives slightly better results than formula B.

If proper Gaussian formulas had been used instead of Simpson's rule in [23] the given accuracy could have been obtained with less than $\frac{1}{8}$

of the number of points used*. Moreover, taking advantage of the full symmetry (as was done in [23]!) of the integrand we could have reduced the number of points by an additional factor of $2^4 = 16$ (an even number of evaluation points is assumed in each one-dimensional integration formula from which we obtain an integration formula over a 4-dimensional cube). Nonetheless, we note the superior accuracy of the integration formulas constructed for the particular regions over which they are used as compared with integration formulas constructed over rectangular regions and used over curvilinear regions.

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* The following procedure is implied here.

First apply the transformation theorem given in the second [redacted] section of Chapter IV. The resulting jacobian of the transformation breaks up into suitable weight functions with respect to which the Gegenbauer (Ultraspherical) polynomials [13] are orthogonal over the interval $(-1,1)$. The numerical integration formulas (repeated sums) resulting from using the zeros of these polynomials as evaluation points can then be used to evaluate the integral. We have applied the method of Chapter V to obtain the above estimate on the number of points required.

APPENDIX B - TABLE V

ZEROS AND WEIGHTS OF FORMULAS OF PRECISION NINE

In this Table we have tabulated zeros U, V $(U = u, V = v)$ and weights A, B, C, D, E, F, H, I, J $(G = 0)$ of the precision nine integration formula(s) given on pages 42-46. The zeros and weights are for the four following regions and weight functions:

Region	Weight Function
N - Cube	1
N - Sphere	1
N - Cube	$1 / \prod_{i=1}^{N} [1 - (x^i)^2]^{\frac{1}{2}}$
Infinite N-Space	$\exp\left[- \sum_{i=1}^{N} (x^i)^2 \right]$

Tabulation of the zeros and weights is carried out for increasing dimension $N (N = n)$ starting from 4.

Only nine out of the twelve equations (page 45) were required to compute the weights. The other three equations were used as checks to compute the errors in the computations. These errors, ER1, ER2, and ER3 are listed following the listing of the zeros and weights. Equation (ix) page 45 was used to compute ER1; equation (v) page 45 was used to compute ER2; and equation (iv) page 45 was used to compute ER3. On comparing the magnitude of the errors and weights we find that the largest of B or C may be in error by 2 in the 19th significant figure. Since, however, the data were obtained by truncating 20 significant figure results to 18 significant

figures we feel that the largest of B or C (and also A) is correct to 18 significant figures unless--unless the offline printer has made errors. When in good running order it is estimated that the offline printer may make one error in printing 100,000 digits. Since the printer was in good running order when the results were printed it may have made two errors. Any other weight R will have at least

$$18 - \log_{10} \frac{\max(B,C)}{R}$$

correct significant figures since the results were obtained in the order given on pages 44 and 46.

Computations were carried out using 20 figure floating point arithmetic on the I.B.M. 1620 at the University of Alberta. The printer referred to above is the I.B.M. 407.

N - CUBE W(X) = 1

N = 4

```
U =  .538469 310105 683091 E-00        U =  .906179 845938 663992 E-00
V =  .906179 845938 663992 E-00        V =  .538469 310105 683091 E-00
A = -.132267 670164 360779 E+02        A = -.352374 470972 661015 E+02
B =  .573058 788563 546878 E+01        B = -.154876 291992 902159 E+01
C = -.137499 093434 472746 E+01        C =  .169096 999116 347747 E+02
D = -.161396 043885 811649 E+01        D =  .683758 023697 763007 E-01
E = -.185101 904223 707624 E-01        E = -.720351 645185 776945 E+01
F =  .453600 000000 000000 E-00        F =  .453600 000000 000000 E-00
H =  .434429 963960 735315 E-01        H =  .279477 800649 982648 E+01
I =  .632575 513938 520240 E-00        I =  .390403 982699 845607 E-01
J =  .173189 000719 477950 E-01        J = -.764813 489311 393001 E-00
```

N = 5

```
U =  .538469 310105 683091 E-00        U =  .906179 845938 663992 E-00
V =  .906179 845938 663992 E-00        V =  .538469 310105 683091 E-00
A = -.793959 568673 149847 E+02        A = -.518826 067662 774552 E+02
B =  .262238 257497 197273 E+02        B = -.456078 267701 739541 E+01
C = -.473455 466260 168953 E+01        C =  .150447 137237 204214 E+02
D = -.575822 293347 031394 E+01        D = -.194099 883403 856417 E-01
E =  .457545 062537 246555 E-01        E = -.156605 592372 057421 E+01
F =  .907200 000000 000000 E-00        F =  .907200 000000 000000 E-00
H = -.217214 981980 367657 E-01        H = -.139738 900324 991324 E+01
I =  .632575 513938 520239 E-00        I =  .390403 982699 845607 E-01
J =  .444707 728194 937522 E-01        J =  .981922 764750 998550 E-00
```

N = 6

```
U =  .538469 310105 683091 E-00        U =  .906179 845938 663992 E-00
V =  .906179 845938 663992 E-00        V =  .538469 310105 683091 E-00
A = -.353559 326955 477331 E+03        A =  .306761 075469 423376 E+03
B =  .954678 957536 921327 E+02        B = -.112152 171985 938246 E+02
C = -.146906 569102 797071 E+02        C = -.128114 344766 293985 E+03
D = -.165770 499784 487898 E+02        D = -.351143 162840 647769 E-00
E =  .691488 750665 116987 E-00        E =  .504976 221775 470457 E+02
F =  .181440 000000 000000 E+01        F =  .181440 000000 000000 E+01
H = -.260657 978376 441189 E-00        H = -.167686 680389 989588 E+02
I =  .843434 018584 693653 E-00        I =  .520538 643599 794143 E-01
J =  .954968 607560 529459 E-01        J =  .363821 202508 452013 E+01
```

N - CUBE   W(X) = 1

N = 7

U =	.538469 310105 683091 E-00	U =	.906179 845938 663992 E-00
V =	.906179 845938 663992 E-00	V =	.538469 310105 683091 E-00
A =	-.132655 593205 210437 E+04	A =	.291049 998350 767516 E+04
B =	.303966 168610 599818 E+03	B =	-.249520 144267 863757 E+02
C =	-.440669 328410 587296 E+02	C =	-.925736 154249 323833 E+03
D =	-.432753 081799 139034 E+02	D =	-.132693 269800 104851 E+01
E =	.345179 690556 703995 E+01	E =	.264150 272535 067009 E+03
F =	.362880 000000 000000 E+01	F =	.362880 000000 000000 E+01
H =	-.955745 920713 617694 E-00	H =	-.614851 161429 961826 E+02
I =	.126515 102787 704048 E+01	I =	.780807 965399 691215 E-01
J =	.197549 036629 171333 E-00	J =	.895079 054575 156330 E+01

N = 8

U =	.538469 310105 683091 E-00	U =	.906179 845938 663992 E-00
V =	.906179 845938 663992 E-00	V =	.538469 310105 683091 E-00
A =	-.444911 507755 310953 E+04	A =	.152724 542748 705916 E+05
B =	.887972 868617 051137 E+03	B =	-.516157 415937 315217 E+02
C =	-.129465 591135 495290 E+03	C =	-.412026 931903 063791 E+04
D =	-.106793 032805 860454 E+03	D =	-.390315 814064 160296 E+01
E =	.127789 524754 506331 E+02	E =	.966401 721690 072914 E+03
F =	.725760 000000 000000 E+01	F =	.725760 000000 000000 E+01
H =	-.278035 176934 870602 E+01	H =	-.178865 792415 988894 E+03
I =	.202424 164460 326476 E+01	I =	.124929 274463 950594 E-00
J =	.402964 451398 821196 E-00	J =	.199108 208862 021542 E+02

N = 9

U =	.538469 310105 683091 E-00	U =	.906179 845938 663992 E-00
V =	.906179 845938 663992 E-00	V =	.538469 310105 683091 E-00
A =	-.137852 396450 258626 E+05	A =	.638684 396801 424606 E+05
B =	.244398 635440 464606 E+04	B =	-.999920 140297 032189 E+02
C =	-.372466 487425 876408 E+03	C =	-.150848 613023 622656 E+05
D =	-.254070 898503 786204 E+03	D =	-.103049 017705 622178 E+02
E =	.407840 619912 202552 E+02	E =	.303258 803714 000973 E+04
F =	.145152 000000 000000 E+02	F =	.145152 000000 000000 E+02
H =	-.729842 339454 035330 E+01	H =	-.469522 705091 970849 E+03
I =	.337373 607433 877461 E+01	I =	.208215 457439 917657 E-00
J =	.816417 406984 947099 E-00	J =	.425006 281653 363453 E+02

N - CUBE W(X) = 1

N = 10

U =	.538469 310105 683091 E-00	U =	.906179 845938 663992 E-00
V =	.906179 845938 663992 E-00	V =	.538469 310105 683091 E-00
A =	-.402681 923756 184877 E+05	A =	.234425 095033 140206 E+06
B =	.643997 305190 806127 E+04	B =	-.180179 300467 732058 E+03
C =	-.104764 905250 452833 E+04	C =	-.493661 541818 804739 E+05
D =	-.589111 462791 702999 E+03	D =	-.256069 745196 824594 E+02
E =	.118971 317486 450253 E+03	E =	.871190 974283 971953 E+04
F =	.290304 000000 000000 E+02	F =	.290304 000000 000000 E+02
H =	-.180722 865007 665891 E+02	H =	-.116262 765070 392781 E+04
I =	.578354 755600 932790 E+01	I =	.356940 784182 715984 E-00
J =	.164781 839138 032949 E+01	J =	.888283 797491 470291 E+02

N = 11

U =	.538469 310105 683091 E-00	U =	.906179 845938 663992 E-00
V =	.906179 845938 663992 E-00	V =	.538469 310105 683091 E-00
A =	-.112390 968057 806059 E+06	A =	.788286 060849 758505 E+06
B =	.164157 850075 290239 E+05	B =	-.294097 567199 805899 E+03
C =	-.287982 471269 410249 E+04	C =	-.149929 401939 340648 E+06
D =	-.134016 225715 166718 E+04	D =	-.612082 909964 809664 E+02
E =	.326650 780827 663523 E+03	E =	.236116 157848 787836 E+05
F =	.580608 000000 000000 E+02	F =	.580608 000000 000000 E+02
H =	-.430954 524249 049433 E+02	H =	-.277241 978244 782787 E+04
I =	.101212 082230 163238 E+02	I =	.624646 372319 752972 E-00
J =	.331811 214887 631192 E+01	J =	.183397 444626 005565 E+03

N = 12

U =	.538469 310105 683091 E-00	U =	.906179 845938 663992 E-00
V =	.906179 845938 663992 E-00	V =	.538469 310105 683091 E-00
A =	-.302521 740776 808677 E+06	A =	.248667 163906 597191 E+07
B =	.407669 242575 145771 E+05	B =	-.402369 909823 676441 E+03
C =	-.774249 858090 426033 E+04	C =	-.431438 767720 696105 E+06
D =	-.300420 317743 985672 E+04	D =	-.142405 265907 194028 E+03
E =	.858521 371058 340139 E+03	E =	.613874 820923 161454 E+05
F =	.116121 600000 000000 E+03	F =	.116121 600000 000000 E+03
H =	-.100092 663696 553416 E+03	H =	-.643916 852697 560021 E+04
I =	.179932 590631 401312 E+02	I =	.111048 243967 956084 E+01
J =	.667118 597837 697288 E+01	J =	.375724 843895 117921 E+03

N - CUBE W(X) = 1

N = 13

U =	.538469 310105 683091 E-00	U =	.906179 845938 663992 E-00
V =	.906179 845938 663992 E-00	V =	.538469 310105 683091 E-00
A =	-.790586 586384 953889 E+06	A =	.746852 492170 505752 E+07
B =	.991319 098700 036654 E+05	B =	-.326483 056286 244329 E+03
C =	-.203895 014239 065069 E+05	C =	-.119156 364820 377219 E+07
D =	-.665616 368115 275817 E+04	D =	-.324787 899642 852246 E+03
E =	.218308 939630 968058 E+04	E =	.154680 781078 069224 E+06
F =	.232243 200000 000000 E+03	F =	.232243 200000 000000 E+03
H =	-.227988 845086 593893 E+03	H =	-.146669 949781 110893 E+05
I =	.323878 663136 522362 E+02	I =	.199886 839142 320951 E+01
J =	.133983 106457 529041 E+02	J =	.765737 615219 206705 E+03

N = 14

U =	.538469 310105 683091 E-00	U =	.906179 845938 663992 E-00
V =	.906179 845938 663992 E-00	V =	.538469 310105 683091 E-00
A =	-.201594 595599 302517 E+07	A =	.215710 509895 057214 E+08
B =	.236914 648190 079258 E+06	B =	.516760 042568 204127 E+03
C =	-.526880 514156 886342 E+05	C =	-.318586 115548 228423 E+07
D =	-.146078 420148 516058 E+05	D =	-.729530 534942 632872 E+03
E =	.540948 617177 931001 E+04	E =	.380327 827639 651874 E+06
F =	.464486 400000 000000 E+03	F =	.464486 400000 000000 E+03
H =	-.511584 725560 161907 E+03	H =	-.329113 058045 419566 E+05
I =	.588870 296611 858841 E+02	I =	.363430 616622 401729 E+01
J =	.268881 573280 495585 E+02	J =	.155485 547532 218391 E+04

N = 15

U =	.538469 310105 683091 E-00	U =	.906179 845938 663992 E-00
V =	.906179 845938 663992 E-00	V =	.538469 310105 683091 E-00
A =	-.503504 007807 209720 E+07	A =	.603419 623252 124721 E+08
B =	.558040 364760 548180 E+06	B =	.379939 765225 474520 E+04
C =	-.133839 136337 583597 E+06	C =	-.829653 182300 056607 E+07
D =	-.318067 133347 953905 E+05	D =	-.161897 054119 912250 E+04
E =	.131280 152434 264142 E+05	E =	.916897 449639 609707 E+06
F =	.928972 800000 000000 E+03	F =	.928972 800000 000000 E+03
H =	-.113438 352189 427205 E+04	H =	-.729772 433057 234691 E+05
I =	.107959 554378 840787 E+03	I =	.666289 463807 736504 E+01
J =	.539288 747170 053673 E+02	J =	.314867 802545 065200 E+04

N - CUBE W(X) = 1

N = 16

U = .538469 310105 683091 E-00			U = .906179 845938 663992 E-00		
V = .906179 845938 663992 E-00			V = .538469 310105 683091 E-00		
A = -.123542 611675 013758 E+08			A = .164351 000230 214986 E+09		
B = .129832 168924 236731 E+07			B = .139834 009524 200670 E+05		
C = -.334783 924994 681616 E+06			C = -.211386 033114 192796 E+08		
D = -.687954 852797 751388 E+05			D = -.355776 002502 595853 E+04		
E = .313189 725696 842097 E+05			E = .217489 701478 638955 E+07		
F = .185794 560000 000000 E+04			F = .185794 560000 000000 E+04		
H = -.249119 518533 644059 E+04			H = -.160263 750004 726049 E+06		
I = .199309 946545 552223 E+03			I = .123007 285626 043662 E+02		
J = .108115 927998 621312 E+03			J = .636330 033134 270798 E+04		

N = 17

U = .538469 310105 683091 E-00	U = .906179 845938 663992 E-00
V = .906179 845938 663992 E-00	V = .538469 310105 683091 E-00
A = -.298504 538256 334508 E+08	A = .437607 087167 815191 E+09
B = .298876 294384 825976 E+07	B = .424417 142280 090926 E+05
C = -.825917 750374 978771 E+06	C = -.528746 013435 133918 E+08
D = -.147955 087779 918993 E+06	D = -.775515 793530 734411 E+04
E = .736535 418712 227681 E+05	E = .508923 531416 023584 E+07
F = .371589 120000 000000 E+04	F = .371589 120000 000000 E+04
H = -.542724 665376 867415 E+04	H = -.349146 026796 010322 E+06
I = .370147 043584 596986 E+03	I = .228442 101876 938229 E+02
J = .216674 447822 289328 E+03	J = .128396 480005 849656 E+05

N = 18

U = .538469 310105 683091 E-00	U = .906179 845938 663992 E-00
V = .906179 845938 663992 E-00	V = .538469 310105 683091 E-00
A = -.711603 920869 237620 E+08	A = .114269 656682 065435 E+10
B = .681704 031026 553630 E+07	B = .117244 655157 051713 E+06
C = -.201237 134327 912836 E+07	C = -.130185 571122 345989 E+09
D = -.316638 410000 575417 E+06	D = -.167895 916411 255423 E+05
E = .171117 702338 537405 E+06	E = .117718 273046 416180 E+08
F = .743178 240000 000000 E+04	F = .743178 240000 000000 E+04
H = -.117442 058737 289342 E+05	H = -.755529 107165 137091 E+06
I = .690941 148024 581040 E+03	I = .426425 256836 951362 E+02
J = .434116 054807 992943 E+03	J = .258752 447201 958174 E+05

N - CUBE W(X) = 1

N = 19

U =	.538469 310105 683091 E-00		U =	.906179 845938 663992 E-00
V =	.906179 845938 663992 E-00		V =	.538469 310105 683091 E-00
A =	-.167633 725590 018681 E+09		A =	.293367 884555 832009 E+10
B =	.154236 600493 530391 E+08		B =	.306034 567825 561705 E+06
C =	-.484860 415566 201051 E+07		C =	-.316184 933531 854404 E+09
D =	-.674733 288882 625697 E+06		D =	-.361377 348232 727928 E+05
E =	.393415 492134 024892 E+06		E =	.269593 161762 179946 E+08
F =	.148635 648000 000000 E+05		F =	.148635 648000 000000 E+05
H =	-.252678 368798 410403 E+05		H =	-.162553 232147 650707 E+07
I =	.129551 465254 608945 E+04		I =	.799547 356569 283804 E+02
J =	.869574 638151 960890 E+03		J =	.520933 996986 869356 E+05

N = 20

U =	.538469 310105 683091 E-00		U =	.906179 845938 663992 E-00
V =	.906179 845938 663992 E-00		V =	.538469 310105 683091 E-00
A =	-.390738 267572 109444 E+09		A =	.742035 553217 184240 E+10
B =	.346475 801237 178771 E+08		B =	.768805 258892 822411 E+06
C =	-.115647 462186 814183 E+08		C =	-.758795 351328 251548 E+09
D =	-.143237 951552 820112 E+07		D =	-.773925 727285 890019 E+05
E =	.896308 859711 482636 E+06		E =	.612078 519148 904377 E+08
F =	.297271 296000 000000 E+05		F =	.297271 296000 000000 E+05
H =	-.540945 240244 484243 E+05		H =	-.348001 285724 547993 E+07
I =	.243861 581655 734485 E+04		I =	.150503 031824 806363 E+03
J =	.174151 844430 858354 E+04		J =	.104791 935147 306754 E+06

N = 21

U =	.538469 310105 683091 E-00		U =	.906179 845938 663992 E-00
V =	.906179 845938 663992 E-00		V =	.538469 310105 683091 E-00
A =	-.902165 070383 616116 E+09		A =	.185227 918789 796325 E+11
B =	.773378 826321 950840 E+08		B =	.187837 398070 660770 E+07
C =	-.273326 689924 955414 E+08		C =	-.180186 906028 036408 E+10
D =	-.303058 490658 230169 E+07		D =	-.165019 351621 264836 E+06
E =	.202580 887136 889634 E+07		E =	.137909 935811 859635 E+09
F =	.594542 592000 000000 E+05		F =	.594542 592000 000000 E+05
H =	-.115306 748578 429536 E+06		H =	-.741792 214307 589144 E+07
I =	.460627 432016 387360 E+04		I =	.284283 504557 967575 E+03
J =	.348724 874284 767691 E+04		J =	.210659 667183 383080 E+06

N - CUBE W(X) = 1

N = 22

U =	.538469	310105	683091	E-00		U =	.906179	845938	663992	E-00
V =	.906179	845938	663992	E-00		V =	.538469	310105	683091	E-00
A =	-.206520	120361	964735	E+10		A =	.456957	065913	059751	E+11
B =	.171645	614703	380291	E+09		B =	.449285	732013	027096	E+07
C =	-.640648	341803	285741	E+08		C =	-.423881	278542	668059	E+10
D =	-.639282	156421	640228	E+07		D =	-.350507	115570	703338	E+06
E =	.454647	084874	778560	E+07		E =	.308639	921302	216518	E+09
F =	.118908	518400	000000	E+06		F =	.118908	518400	000000	E+06
H =	-.244848	898215	924446	E+06		H =	-.157516	371433	216460	E+08
I =	.872767	765925	786577	E+04		I =	.538642	429688	780668	E+03
J =	.698203	448800	258190	E+04		J =	.423244	444588	774866	E+06

N = 23

U =	.538469	310105	683091	E-00		U =	.906179	845938	663992	E-00
V =	.906179	845938	663992	E-00		V =	.538469	310105	683091	E-00
A =	-.469090	512620	065029	E+10		A =	.111545	815821	217172	E+12
B =	.378999	737623	683756	E+09		B =	.105670	982234	449125	E+08
C =	-.149028	830129	948817	E+09		C =	-.988813	714267	908589	E+10
D =	-.134489	466305	364023	E+08		D =	-.741951	055797	754008	E+06
E =	.101395	895137	518185	E+08		E =	.686583	113390	106983	E+09
F =	.237817	036800	000000	E+06		F =	.237817	036800	000000	E+06
H =	-.518168	598549	979643	E+06		H =	-.333348	600009	830183	E+08
I =	.165825	875525	899449	E+05		I =	.102342	061640	868327	E+04
J =	.139776	355801	581743	E+05		J =	.849954	087577	165404	E+06

N = 24

U =	.538469	310105	683091	E-00		U =	.906179	845938	663992	E-00
V =	.906179	845938	663992	E-00		V =	.538469	310105	683091	E-00
A =	-.105794	265610	217403	E+11		A =	.269702	525362	422178	E+12
B =	.832954	110359	099716	E+09		B =	.245152	933447	590855	E+08
C =	-.344284	088972	659392	E+09		C =	-.228926	745989	119786	E+11
D =	-.282245	002652	800003	E+08		D =	-.156577	576090	820267	E+07
E =	.224863	578684	886546	E+08		E =	.151909	911120	892076	E+10
F =	.475634	073600	000000	E+06		F =	.475634	073600	000000	E+06
H =	-.109327	880133	622078	E+07		H =	-.703328	914306	454892	E+08
I =	.315858	810525	522761	E+05		I =	.194937	260268	320622	E+04
J =	.279798	202535	456059	E+05		J =	.170617	853416	315916	E+07

N - CUBE W(X) = 1

N = 25

U =	.538469	310105	683091	E-00	U =	.906179	845938	663992	E-00
V =	.906179	845938	663992	E-00	V =	.538469	310105	683091	E-00
A =	-.237045	757833	813934	E+11	A =	.646477	174791	141234	E+12
B =	.182289	272829	743554	E+10	B =	.562294	399482	577301	E+08
C =	-.790332	778553	089232	E+09	C =	-.526382	095361	657216	E+11
D =	-.591022	145389	743918	E+08	D =	-.329529	882044	179468	E+07
E =	.496148	398358	923904	E+08	E =	.334471	667698	997293	E+10
F =	.951268	147200	000000	E+06	F =	.951268	147200	000000	E+06
H =	-.230044	081114	496457	E+07	H =	-.147992	125718	649883	E+09
I =	.603003	183730	543453	E+05	I =	.372152	951421	339371	E+04
J =	.560042	752220	534976	E+05	J =	.342375	772106	964261	E+07

N - CUBE W(X) = 1

ERRORS

N				
N = 4	ER1 =	.80000E-19	ER1 =	.60000E-19
	ER2 =	.30000E-18	ER2 =	.28000E-18
	ER3 =	.30000E-18	ER3 =	.25000E-18
N = 5	ER1 =	.10000E-18	ER1 =	.13000E-18
	ER2 =	.50000E-18	ER2 =	.48000E-18
	ER3 =	.50000E-18	ER3 =	.44000E-18
N = 6	ER1 =	.00000E-99	ER1 =	.30000E-18
	ER2 =	.23000E-17	ER2 =	.13500E-17
	ER3 =	.25000E-17	ER3 =	.13700E-17
N = 7	ER1 =	.40000E-18	ER1 =	.40000E-18
	ER2 =	.40000E-17	ER2 =	.49900E-17
	ER3 =	.40000E-17	ER3 =	.88000E-17
N = 8	ER1 =	.15000E-17	ER1 =	.70000E-18
	ER2 =	.13000E-16	ER2 =	.87200E-17
	ER3 =	.15000E-16	ER3 =	.12800E-16
N = 9	ER1 =	-.10000E-16	ER1 =	.10000E-16
	ER2 =	.80000E-17	ER2 =	.26000E-16
	ER3 =	.10000E-16	ER3 =	.54800E-16
N = 10	ER1 =	-.11000E-16	ER1 =	-.21000E-16
	ER2 =	.50000E-16	ER2 =	.33700E-16
	ER3 =	.70000E-16	ER3 =	.57000E-16
N = 11	ER1 =	-.27000E-16	ER1 =	-.50000E-17
	ER2 =	.11000E-15	ER2 =	.36840E-15
	ER3 =	.12000E-15	ER3 =	.11670E-14
N = 12	ER1 =	-.26900E-15	ER1 =	.31000E-16
	ER2 =	.12000E-15	ER2 =	.17200E-15
	ER3 =	.12000E-15	ER3 =	.38300E-15
N = 13	ER1 =	.43000E-15	ER1 =	-.26000E-15
	ER2 =	.53500E-14	ER2 =	.86800E-15
	ER3 =	.66000E-14	ER3 =	.27350E-14
N = 14	ER1 =	.30000E-16	ER1 =	.10500E-14
	ER2 =	.80000E-15	ER2 =	.31500E-14
	ER3 =	.90000E-15	ER3 =	.10250E-13

N - CUBE W(X) = 1

ERRORS

N = 15

ER1 = .16000E-15 ER1 = .20000E-15
ER2 = .18000E-14 ER2 = .15500E-14
ER3 = .33000E-14 ER3 = .20900E-14

N = 16

ER1 = -.19000E-14 ER1 = -.39000E-14
ER2 = .27000E-14 ER2 = .74500E-14
ER3 = .37000E-14 ER3 = .22080E-13

N = 17

ER1 = -.15800E-13 ER1 = -.57000E-14
ER2 = .10000E-13 ER2 = .62700E-13
ER3 = .22000E-13 ER3 = .19750E-12

N = 18

ER1 = -.60100E-13 ER1 = .10100E-13
ER2 = .84000E-13 ER2 = .42580E-13
ER3 = .10300E-12 ER3 = .10340E-12

N = 19

ER1 = -.20000E-14 ER1 = .38000E-13
ER2 = .73000E-12 ER2 = .94200E-13
ER3 = .93000E-12 ER3 = .21770E-12

N = 20

ER1 = -.16500E-12 ER1 = -.65000E-13
ER2 = .41000E-12 ER2 = .13612E-11
ER3 = .63000E-12 ER3 = .43840E-11

N = 21

ER1 = -.63100E-12 ER1 = .16900E-12
ER2 = .73000E-12 ER2 = .74458E-11
ER3 = .10100E-11 ER3 = .24628E-10

N = 22

ER1 = -.56200E-12 ER1 = -.26200E-12
ER2 = .77800E-11 ER2 = .13600E-11
ER3 = .96800E-11 ER3 = .43790E-11

N = 23

ER1 = .87000E-12 ER1 = -.11200E-11
ER2 = .45200E-11 ER2 = .18660E-11
ER3 = .56000E-11 ER3 = .45810E-11

N = 24

ER1 = -.32600E-11 ER1 = -.24000E-12
ER2 = .98000E-11 ER2 = .84650E-11
ER3 = .13600E-10 ER3 = .27470E-10

N = 25

ER1 = -.12300E-10 ER1 = -.72600E-11
ER2 = .10100E-10 ER2 = .65000E-11
ER3 = .15100E-10 ER3 = .10680E-10

N - SPHERE W(X) = 1

N = 4

U = .442930 458136 057122 E-00		U = .798214 220988 774342 E-00		
V = .798214 220988 774342 E-00		V = .442930 458136 057122 E-00		
A = -.101782 549866 074272 E-00		A = -.124801 108358 880783 E+01		
B = .911474 398556 227284 E-01		B = -.424086 526897 493522 E-01		
C = -.372099 907247 185065 E-01		C = .669460 368682 020355 E-00		
D = .400121 069994 910212 E-01		D = -.258768 113998 110770 E-02		
E = -.518701 212249 653052 E-02		E = -.249144 357413 707792 E-00		
F = .528728 807201 215640 E-01		F = .528728 807201 215640 E-01		
H = .129966 549125 771141 E-02		H = .144578 232206 599406 E-00		
I = .747088 753316 896630 E-01		I = .303164 290787 329538 E-03		
J = -.346668 454841 526167 E-03		J = .241975 922838 995968 E-02		

N = 5

U = .420914 805023 811444 E-00	U = .769455 324331 787326 E-00
V = .769455 324331 787326 E-00	V = .420914 805023 811444 E-00
A = -.147776 465838 512509 E+01	A = -.294842 340301 736794 E+01
B = .513986 808405 460051 E-00	B = -.133421 980109 643685 E-00
C = -.128666 935307 319821 E-00	C = .110700 535106 068105 E+01
D = -.960356 695563 911229 E-01	D = -.257334 090599 197498 E-02
E = -.435648 270686 342397 E-02	E = -.318417 623052 098999 E-00
F = .485386 100042 992531 E-01	F = .485386 100042 992531 E-01
H = .594380 600290 482999 E-03	H = .741273 178319 026253 E-01
I = .371683 618407 830687 E-01	I = .147755 602802 681829 E-03
J = -.839547 269938 920242 E-06	J = .186365 323828 074123 E-01

N = 6

U = .401905 036089 210065 E-00	U = .743477 004521 219230 E-00
V = .743477 004521 219230 E-00	V = .401905 036089 210065 E-00
A = -.434087 873524 197482 E+01	A = -.434087 873524 197482 E+01
B = .120766 407050 434449 E+01	B = -.197089 126850 128130 E-00
C = -.197089 126850 128130 E-00	C = .120766 407050 434449 E+01
D = -.217959 756492 935725 E-00	D = -.252932 212252 966264 E-02
E = -.252932 212252 966264 E-02	E = -.217959 756492 935725 E-00
F = .422029 877037 414219 E-01	F = .422029 877037 414219 E-01
H = .133896 643114 035872 E-21	H = .183621 033568 859641 E-19
I = .230216 158590 861082 E-01	I = .895440 739151 465994 E-04
J = .895440 739151 465994 E-04	J = .230216 158590 861082 E-01

N - SPHERE W(X) = 1

N = 7

U = .385270 382880 547445 E-00	U = .719884 295384 850745 E-00
V = .719884 295384 850745 E-00	V = .385270 382880 547445 E-00
A = -.853984 718832 534907 E+01	A = -.415796 381192 531169 E+01
B = .199595 992945 675215 E+01	B = -.228900 991731 237094 E-00
C = -.237383 980846 572939 E-00	C = .735510 261369 977054 E-00
D = -.304649 099939 537498 E-00	D = -.240324 148253 265722 E-02
E = -.282494 203698 695919 E-03	E = .104633 170821 562761 E-01
F = .348988 395535 842131 E-01	F = .348988 395535 842131 E-01
H = -.424149 455766 792260 E-03	H = -.630224 834043 387549 E-01
I = .150940 828799 686371 E-01	I = .574709 179866 488169 E-04
J = .110489 599957 497849 E-03	J = .229718 933055 109815 E-01

N = 8

U = .370551 990877 550432 E-00	U = .698348 925721 722422 E-00
V = .698348 925721 722422 E-00	V = .370551 990877 550432 E-00
A = -.133424 630785 895472 E+02	A = -.152331 334650 909076 E+01
B = .268420 057590 213119 E+01	B = -.231736 300404 078068 E-00
C = -.250420 619233 127021 E-00	C = -.289271 175947 031886 E-00
D = -.347548 646320 032048 E-00	D = -.218388 142157 715042 E-02
E = .181990 118464 762534 E-02	E = .289623 871933 360040 E-00
F = .275500 459490 107903 E-01	F = .275500 459490 107903 E-01
H = -.667297 101037 462628 E-03	H = -.106195 419708 898681 E-00
I = .999423 888059 915408 E-02	I = .372940 540753 767369 E-04
J = .104023 764179 122999 E-03	J = .206137 808514 890222 E-01

N = 9

U = .357406 744336 593255 E-00	U = .678598 344545 847034 E-00
V = .678598 344545 847034 E-00	V = .357406 744336 593255 E-00
A = -.177498 042074 907983 E+02	A = .352673 151227 556969 E+01
B = .312937 643188 598842 E+01	B = -.213002 476620 740139 E-00
C = -.241164 711199 316142 E-00	C = -.162690 485152 955825 E+01
D = -.349440 307868 431467 E-00	D = -.189125 664093 167268 E-02
E = .338916 234255 132784 E-02	E = .542362 432771 983535 E-00
F = .208319 290404 608016 E-01	F = .208319 290404 608016 E-01
H = -.754345 569069 000075 E-03	H = -.127400 391520 059286 E-00
I = .655953 965735 879034 E-02	I = .240226 922216 706266 E-04
J = .868848 229774 206329 E-04	J = .171762 389506 970641 E-01

N - SPHERE W(X) = 1

N = 10

U =	.345572 105134 864650 E-00	U =	.660405 538823 088315 E-00
V =	.660405 538823 088315 E-00	V =	.345572 105134 864650 E-00
A =	-.208822 605313 513173 E+02	A =	.100413 397886 769629 E+02
B =	.327305 308310 589367 E+01	B =	-.181552 634340 299025 E-00
C =	-.216514 285353 194606 E-00	C =	-.294662 863191 265797 E+01
D =	-.320331 837991 885333 E-00	D =	-.156106 971424 439113 E-02
E =	.426587 212123 820570 E-02	E =	.716281 781177 873274 E-00
F =	.151328 415553 161158 E-01	F =	.151328 415553 161158 E-01
H =	-.728367 729435 324604 E-03	H =	-.129576 702396 219825 E-00
I =	.423182 876475 565346 E-02	I =	.152320 323189 883557 E-04
J =	.672582 987072 258274 E-04	J =	.134873 075073 427838 E-01

N = 11

U =	.334843 229938 888741 E-00	U =	.643580 454051 998609 E-00
V =	.643580 454051 998609 E-00	V =	.334843 229938 888741 E-00
A =	-.222479 827285 205812 E+02	A =	.165841 632433 942430 E+02
B =	.313662 631911 792640 E+01	B =	-.145356 439263 750309 E-00
C =	-.183469 709238 767721 E-00	C =	-.396187 712756 887724 E+01
D =	-.272808 570626 125197 E-00	D =	-.123038 945125 021698 E-02
E =	.448660 104500 239477 E-02	E =	.791966 946376 895349 E-00
F =	.105885 947873 771225 E-01	F =	.105885 947873 771225 E-01
H =	-.635221 166250 290195 E-03	H =	-.118308 390778 113394 E-00
I =	.267298 587270 835376 E-02	I =	.946900 353366 529075 E-05
J =	.491703 264243 084279 E-04	J =	.100672 602963 404409 E-01

N = 12

U =	.325057 583671 868143 E-00	U =	.627963 030199 554375 E-00
V =	.627963 030199 554375 E-00	V =	.325057 583671 868143 E-00
A =	-.218153 997582 550032 E+02	A =	.217963 245758 427948 E+02
B =	.279350 979717 849728 E+01	B =	-.110267 471194 512677 E-00
C =	-.147929 724369 817278 E-00	C =	-.451277 317834 644034 E+01
D =	-.218436 283162 929674 E-00	D =	-.928314 577467 171152 E-03
E =	.420744 721916 527435 E-02	E =	.777875 031681 380001 E-00
F =	.715319 340457 815801 E-02	F =	.715319 340457 815801 E-02
H =	-.513576 179663 244550 E-03	H =	-.996311 314844 309676 E-01
I =	.165007 155109 025981 E-02	I =	.576025 552505 453614 E-05
J =	.342922 655063 459000 E-04	J =	.718513 441133 642469 E-02

N - SPHERE W(X) = 1

N = 13

U =	.316084 301561 041006	E-00			U =	.613417 631387 576827	E-00
V =	.613417 631387 576827	E-00			V =	.316084 301561 041006	E-00
A =	-.199197 125930 941044	E+02			A =	.248150 399828 053347	E+02
B =	.233577 045859 183191	E+01			B =	-.797574 871954 996592	E-01
C =	-.114134 566126 284454	E-00			C =	-.458087 624740 040505	E+01
D =	-.165803 227059 994440	E-00			D =	-.672249 472532 476594	E-03
E =	.362488 539381 562282	E-02			E =	.698777 611189 035181	E-00
F =	.467534 211784 873130	E-02			F =	.467534 211784 873130	E-02
H =	-.390648 624213 463583	E-03			H =	-.785982 580226 390565	E-01
I =	.994859 468533 927365	E-03			I =	.342639 695852 366695	E-05
J =	.229588 281691 968461	E-04			J =	.492477 236966 588019	E-02

N = 14

U =	.307816 655330 686165	E-00	U =	.599828 609893 262845	E-00
V =	.599828 609893 262845	E-00	V =	.307816 655330 686165	E-00
A =	-.170881 494626 012242	E+02	A =	.254201 758106 493917	E+02
B =	.184763 807380 216358	E+01	B =	-.552699 260229 385589	E-01
C =	-.846186 576219 726305	E-01	C =	-.425432 855397 552991	E+01
D =	-.120037 247074 189256	E-00	D =	-.468452 394893 768640	E-03
E =	.291793 971268 708577	E-02	E =	.584035 825361 698454	E-00
F =	.296205 098234 541835	E-02	F =	.296205 098234 541835	E-02
H =	-.282199 342298 404534	E-03	H =	-.586727 560363 239759	E-01
I =	.585810 091047 998289	E-03	I =	.199263 540652 910461	E-05
J =	.148198 782382 747652	E-04	J =	.325275 354724 454265	E-02

N = 15

U =	.300166 604546 498365	E-00	U =	.587096 770737 736421	E-00
V =	.587096 770737 736421	E-00	V =	.300166 604546 498365	E-00
A =	-.138676 300641 454082	E+02	A =	.239368 084647 894069	E+02
B =	.139088 040548 303482	E+01	B =	-.368341 756692 429392	E-01
C =	-.604789 642251 292222	E-01	C =	-.367335 618693 082680	E+01
D =	-.832808 380326 598527	E-01	D =	-.314877 884614 655203	E-03
E =	.221849 231780 173225	E-02	E =	.459315 939725 968179	E-00
F =	.182209 038814 331333	E-02	F =	.182209 038814 331333	E-02
H =	-.194874 630955 106727	E-03	H =	-.417382 136737 406178	E-01
I =	.336992 601633 551668	E-03	I =	.113318 242958 996901	E-05
J =	.925295 871938 608267	E-05	J =	.207608 483803 941074	E-02

N - SPHERE W(X) = 1

N = 16

U =	.293060	782359	628900	E-00	U =	.575136	544230	523478	E-00
V =	.575136	544230	523478	E-00	V =	.293060	782359	628900	E-00
A =	-.107056	492134	383927	E+02	A =	.210068	338188	192765	E+02
B =	.100118	473052	222183	E+01	B =	-.236813	352174	792649	E-01
C =	-.417785	764215	555920	E-01	C =	-.298097	288971	406315	E+01
D =	-.555803	255665	118800	E-01	D =	-.204604	123778	962686	E-03
E =	.160511	999662	867002	E-02	E =	.342635	436457	116619	E-00
F =	.108995	239324	118957	E-02	F =	.108995	239324	118957	E-02
H =	-.129266	008600	545193	E-03	H =	-.284439	830016	877499	E-01
I =	.189481	571190	662454	E-03	I =	.630416	361061	336778	E-06
J =	.560218	592262	076730	E-05	J =	.128348	091740	942206	E-02

N = 17

U =	.286437	488950	176166	E-00	U =	.563873	713630	913732	E-00
V =	.563873	713630	913732	E-00	V =	.286437	488950	176166	E-00
A =	-.789769	314402	400580	E+01	A =	.173483	330085	410120	E+02
B =	.691802	930371	112991	E-00	B =	-.147258	858865	313416	E-01
C =	-.279536	993454	621014	E-01	C =	-.229154	560691	899633	E+01
D =	-.357924	976265	778725	E-01	D =	-.128779	780715	655461	E-03
E =	.111132	773105	910326	E-02	E =	.243896	427744	369877	E-00
F =	.634900	818275	023079	E-03	F =	.634900	818275	023079	E-03
H =	-.826738	341183	172484	E-04	H =	-.186459	283580	631833	E-01
I =	.104197	015468	344585	E-03	I =	.343267	413765	034470	E-06
J =	.329590	434656	207905	E-05	J =	.770123	028256	315418	E-03

N = 18

U =	.280244	406274	388118	E-00	U =	.553243	578695	044872	E-00
V =	.553243	578695	044872	E-00	V =	.280244	406274	388118	E-00
A =	-.558867	816938	684210	E+01	A =	.135793	612703	616291	E+02
B =	.460370	083384	416168	E-00	B =	-.887637	641532	545238	E-02
C =	-.181486	535558	054141	E-01	C =	-.167868	435372	811616	E+01
D =	-.222991	649632	406846	E-01	D =	-.786547	452661	429191	E-04
E =	.739487	355364	441942	E-03	E =	.166440	932429	041579	E-00
F =	.360588	437331	965471	E-03	F =	.360588	437331	965471	E-03
H =	-.511338	812894	115538	E-04	H =	-.117962	560870	176415	E-01
I =	.560744	392666	083133	E-04	I =	.183046	715558	291849	E-06
J =	.188750	942520	534364	E-05	J =	.449282	975500	529697	E-03

N - SPHERE W(X) = 1

N = 19

U =	.274436 837849 719332 E-00				U =	.543189 462712 060743 E-00		
V =	.543189 462712 060743 E-00				V =	.274436 837849 719332 E-00		
A =	-.380567 641600 786953 E+01				A =	.101309 110290 372475 E+02		
B =	.295860 087760 626023 E-00				B =	-.519633 797526 951991 E-02		
C =	-.114510 759467 417734 E-01				C =	-.117740 385495 243749 E+01		
D =	-.134702 471184 106554 E-01				D =	-.466932 409591 784160 E-04		
E =	.474534 923330 176044 E-03				E =	.109301 748107 677971 E-00		
F =	.199905 580362 639692 E-03				F =	.199905 580362 639692 E-03		
H =	-.306604 802523 149682 E-04				H =	-.722188 207212 286040 E-02		
I =	.295517 323993 018905 E-04				I =	.956474 348743 320719 E-07		
J =	.105378 744275 917483 E-05				J =	.255235 547153 141278 E-03		

N = 20

U =	.268976 336542 610175 E-00	U =	.533661 491512 151036 E-00
V =	.533661 491512 151036 E-00	V =	.268976 336542 610175 E-00
A =	-.250070 188810 179886 E+01	A =	.723614 191312 675591 E+01
B =	.184055 913901 813496 E-00	B =	-.295931 201827 240778 E-02
C =	-.703139 259000 168886 E-02	C =	-.793700 546792 771262 E-00
D =	-.790473 630087 402368 E-02	D =	-.269821 112668 808651 E-04
E =	.294497 933869 641325 E-03	E =	.692865 632276 458257 E-01
F =	.108293 740549 663743 E-03	F =	.108293 740549 663743 E-03
H =	-.178600 025075 845661 E-04	H =	-.428840 552936 221385 E-02
I =	.152614 327744 930328 E-04	I =	.490034 672389 171784 E-07
J =	.574297 658638 463242 E-06	J =	.141391 007167 499322 E-03

N = 21

U =	.263829 623052 472386 E-00	U =	.524615 588985 770496 E-00
V =	.524615 588985 770496 E-00	V =	.263829 623052 472386 E-00
A =	-.158940 075662 562368 E+01	A =	.496649 651193 645361 E+01
B =	.111068 773146 390682 E-00	B =	-.164193 358122 367365 E-02
C =	-.420690 321387 784018 E-02	C =	-.515867 364920 747035 E-00
D =	-.451392 484662 482540 E-02	D =	-.151975 078641 858198 E-04
E =	.177175 214584 876670 E-03	E =	.425062 855084 105034 E-01
F =	.573810 431771 086727 E-04	F =	.573810 431771 086727 E-04
H =	-.101248 801288 980257 E-04	H =	-.247474 791342 291204 E-02
I =	.772826 651283 978586 E-05	I =	.246308 521740 604391 E-07
J =	.305877 522421 227822 E-06	J =	.764712 641079 206760 E-04

N - SPHERE W(X) = 1

N = 22

U =	.258967	724930	090847	E-00	U =	.516012	645947	622233	E-00
V =	.516012	645947	622233	E-00	V =	.258967	724930	090847	E-00
A =	-.979154	017633	843149	E-00	A =	.328559	385374	730494	E+01
B =	.651325	535879	960435	E-01	B =	-.888713	709416	179242	E-03
C =	-.245522	768072	786726	E-02	C =	-.324138	312327	056380	E-00
D =	-.251201	182570	840959	E-02	D =	-.835358	258487	001183	E-05
E =	.103540	272508	821990	E-03	E =	.252930	500253	667635	E-01
F =	.297652	018961	541089	E-04	F =	.297652	018961	541089	E-04
H =	-.559469	275468	460009	E-05	H =	-.139025	309255	375865	E-02
I =	.383985	226874	345121	E-05	I =	.121529	582333	909009	E-07
J =	.159381	714935	617219	E-06	J =	.404254	599675	265738	E-04

N = 23

U =	.254365	284829	507338	E-00	U =	.507817	828591	015428	E-00
V =	.507817	828591	015428	E-00	V =	.254365	284829	507338	E-00
A =	-.585750	621484	099555	E-00	A =	.210056	272932	018985	E+01
B =	.371760	292179	012156	E-01	B =	-.469808	912942	117712	E-03
C =	-.139916	558855	830155	E-02	C =	-.197345	792745	044481	E-00
D =	-.136415	737052	078804	E-02	D =	-.448601	124348	915476	E-05
E =	.588792	166394	324705	E-04	E =	.146259	668542	255095	E-01
F =	.151280	887243	848817	E-04	F =	.151280	887243	848817	E-04
H =	-.301739	180394	864882	E-05	H =	-.761434	486892	680836	E-03
I =	.187306	729580	942435	E-05	I =	.588948	270351	195284	E-08
J =	.813242	778022	281734	E-07	J =	.209089	294681	264452	E-04

N = 24

U =	.250000	000000	000000	E-00	U =	.500000	000000	000000	E-00
V =	.500000	000000	000000	E-00	V =	.250000	000000	000000	E-00
A =	-.340822	194557	945238	E-00	A =	.130075	784813	911975	E+01
B =	.206826	491707	210896	E-01	B =	-.242826	364048	715102	E-03
C =	-.779290	430289	586015	E-03	C =	-.116652	151786	941864	E-00
D =	-.723767	066970	555751	E-03	D =	-.235601	258779	477783	E-05
E =	.326307	743409	576730	E-04	E =	.823285	038679	007166	E-02
F =	.753924	028094	328907	E-05	F =	.753924	028094	328907	E-05
H =	-.159030	849676	147503	E-05	H =	-.407118	975170	937609	E-03
I =	.897528	604874	201080	E-06	I =	.280477	689023	187837	E-08
J =	.406692	649083	622364	E-07	J =	.105908	375375	155727	E-04

N - SPHERE W(X) = 1

N = 25

U =	.245852 164547 102720 E-00		U =	.492531 233748 487628 E-00
V =	.492531 233748 487628 E-00		V =	.245852 164547 102720 E-00
A =	-.193166 029015 309165 E-00		A =	.781721 161520 260848 E-00
B =	.112300 989235 612540 E-01		B =	-.122830 271941 276963 E-03
C =	-.424577 629042 384056 E-03		C =	-.670626 236763 854542 E-01
D =	-.375569 991215 811434 E-03		D =	-.121124 188406 411297 E-05
E =	.176479 679347 550803 E-04		E =	.451772 517128 085782 E-02
F =	.368679 880490 119195 E-05		F =	.368679 880490 119195 E-05
H =	-.819965 644296 486668 E-06		H =	-.212751 963586 811707 E-03
I =	.422705 080048 475494 E-06		I =	.131331 617180 672226 E-08
J =	.199488 989967 268738 E-07		J =	.525797 697974 874156 E-05

N - SPHERE   W(X) = 1

ERRORS

	N = 4	
ER1 = -.10000E-20		ER1 = -.10000E-20
ER2 = .00000E-99		ER2 = -.30000E-21
ER3 = .10000E-19		ER3 = .90000E-20
	N = 5	
ER1 = .90000E-21		ER1 = .11000E-20
ER2 = .13000E-19		ER2 = .12400E-19
ER3 = .80000E-20		ER3 = .60000E-20
	N = 6	
ER1 = .50000E-21		ER1 = .20000E-21
ER2 = .11000E-19		ER2 = .10400E-19
ER3 = .13000E-19		ER3 = .12000E-19
	N = 7	
ER1 = .20000E-21		ER1 = .10000E-21
ER2 = .20000E-20		ER2 = .14000E-20
ER3 = .40000E-20		ER3 = .33000E-20
	N = 8	
ER1 = .70000E-21		ER1 = .60000E-21
ER2 = .70000E-20		ER2 = .40500E-20
ER3 = .11000E-19		ER3 = .44000E-20
	N = 9	
ER1 = .00000E-99		ER1 = .10000E-21
ER2 = .10000E-20		ER2 = .97000E-21
ER3 = .30000E-20		ER3 = .25000E-20
	N = 10	
ER1 = -.21000E-21		ER1 = .50000E-22
ER2 = .16000E-20		ER2 = .14900E-20
ER3 = .50000E-20		ER3 = .38000E-20
	N = 11	
ER1 = .40000E-22		ER1 = -.60000E-22
ER2 = .12000E-20		ER2 = .11200E-20
ER3 = .19000E-20		ER3 = .19000E-20
	N = 12	
ER1 = .00000E-99		ER1 = .20000E-22
ER2 = .50000E-21		ER2 = .37900E-21
ER3 = .12000E-20		ER3 = .66000E-21
	N = 13	
ER1 = .40000E-22		ER1 = .40000E-22
ER2 = .77000E-21		ER2 = .70700E-21
ER3 = .22000E-20		ER3 = .15300E-20
	N = 14	
ER1 = .26000E-22		ER1 = .60000E-23
ER2 = .27000E-21		ER2 = .25800E-21
ER3 = .70000E-21		ER3 = .10700E-20

N - SPHERE W(X) = 1

ERRORS

N = 15

ER1 = -.90000E-23	ER1 = -.19000E-22
ER2 = .90000E-22	ER2 = .13100E-21
ER3 = .22000E-21	ER3 = .39000E-21

N = 16

ER1 = -.12000E-22	ER1 = .60000E-23
ER2 = .40000E-22	ER2 = .32100E-22
ER3 = .10000E-21	ER3 = .93000E-22

N = 17

ER1 = .70000E-23	ER1 = -.30000E-23
ER2 = .36000E-22	ER2 = .31600E-22
ER3 = .15000E-21	ER3 = .10600E-21

N = 18

ER1 = -.15000E-23	ER1 = -.25000E-23
ER2 = .78000E-22	ER2 = .22600E-22
ER3 = .27200E-21	ER3 = .71000E-22

N = 19

ER1 = -.80000E-24	ER1 = .20000E-24
ER2 = .23000E-22	ER2 = .87300E-23
ER3 = .68000E-22	ER3 = .33700E-22

N = 20

ER1 = -.80000E-25	ER1 = -.15000E-24
ER2 = .24000E-23	ER2 = .56200E-23
ER3 = .13000E-22	ER3 = .58100E-22

N = 21

ER1 = -.32000E-24	ER1 = -.20000E-25
ER2 = .11600E-22	ER2 = .73600E-23
ER3 = .44000E-22	ER3 = .68900E-22

N = 22

ER1 = .15000E-24	ER1 = -.70000E-25
ER2 = .11000E-23	ER2 = .12540E-23
ER3 = .37000E-23	ER3 = .61000E-23

N = 23

ER1 = .77000E-25	ER1 = .77000E-25
ER2 = .61000E-24	ER2 = .34800E-24
ER3 = .46000E-23	ER3 = .35000E-23

N = 24

ER1 = -.27000E-25	ER1 = -.17000E-25
ER2 = .80000E-25	ER2 = .54100E-24
ER3 = .90000E-24	ER3 = .74700E-23

N = 25

ER1 = -.20000E-26	ER1 = -.20000E-26
ER2 = .80000E-25	ER2 = .46800E-25
ER3 = .91000E-24	ER3 = .46700E-24

N - CUBE $\quad W(X) = 1/\prod_{i=1}^{N} [1-(x^i)^2]^{\frac{1}{2}}$

N = 4

U =	.587785 252292 473129 E-00	U =	.951056 516295 153572 E-00
V =	.951056 516295 153572 E-00	V =	.587785 252292 473129 E-00
A =	-.125335 047234 568282 E+03	A =	-.334435 862445 923221 E+03
B =	.543319 540753 596439 E+02	B =	-.153683 176617 586690 E+02
C =	-.130944 304855 560389 E+02	C =	.161156 248857 239743 E+03
D =	-.165052 612498 599841 E+02	D =	.919806 684419 594142 E-00
E =	-.217136 903681 720914 E-00	E =	-.699174 086408 000338 E+02
F =	.389636 364136 009749 E+01	F =	.389636 364136 009749 E+01
H =	.568471 794050 657528 E-00	H =	.267060 736954 700248 E+02
I =	.510040 622280 502039 E+01	I =	.744139 239235 125836 E-00
J =	.459903 342209 797072 E-00	J =	-.825263 062492 999204 E+01

N = 5

U =	.587785 252292 473129 E-00	U =	.951056 516295 153572 E-00
V =	.951056 516295 153572 E-00	V =	.587785 252292 473129 E-00
A =	-.112715 666594 481840 E+04	A =	.515117 296374 255502 E+03
B =	.378100 097730 889543 E+03	B =	-.598396 255541 968327 E+02
C =	-.741268 800498 972777 E+02	C =	-.293096 741692 440465 E+03
D =	-.838996 049279 162512 E+02	D =	-.178590 681196 255562 E+01
E =	.357181 362392 511124 E+01	E =	.167799 209855 832502 E+03
F =	.122407 873914 112581 E+02	F =	.122407 873914 112581 E+02
H =	-.178590 681196 255562 E+01	H =	-.838996 049279 162512 E+02
I =	.801169 935994 395910 E+01	I =	.116889 118361 448447 E+01
J =	.161536 788660 512338 E+01	J =	.289866 005919 230219 E+02

N = 6

U =	.587785 252292 473129 E-00	U =	.951056 516295 153572 E-00
V =	.951056 516295 153572 E-00	V =	.587785 252292 473129 E-00
A =	-.759467 605829 271418 E+04	A =	.233614 588329 277069 E+05
B =	.210791 270806 494966 E+04	B =	-.185134 320914 340791 E+03
C =	-.409557 989732 642111 E+03	C =	-.843522 259116 015870 E+04
D =	-.364256 365888 507671 E+03	D =	-.202993 115416 141031 E+02
E =	.470277 891038 762929 E+02	E =	.279868 422387 902483 E+04
F =	.384555 677430 121774 E+02	F =	.384555 677430 121774 E+02
H =	-.168317 751613 725990 E+02	H =	-.790735 147441 883127 E+03
I =	.167796 639013 133270 E+02	I =	.244811 997019 276169 E+01
J =	.525341 583042 152820 E+01	J =	.148568 855141 627181 E+03

N - CUBE $\qquad W(X) = 1/\prod_{i=1}^{N}[1-(x^i)^2]^{\frac{1}{2}}$

N = 7

U =	.587785	252292	473129	E-00	U =	.951056	516295	153572	E-00
V =	.951056	516295	153572	E-00	V =	.587785	252292	473129	E-00
A =	-.432428	294004	799315	E+05	A =	.240407	571309	748917	E+06
B =	.103561	456689	868694	E+05	B =	-.449583	881878	991474	E+03
C =	-.221220	325500	710193	E+04	C =	-.724494	453356	351979	E+05
D =	-.146063	433615	108224	E+04	D =	-.109918	142292	849629	E+03
E =	.330736	700989	177984	E+03	E =	.192407	634150	044345	E+05
F =	.120811	729111	071682	E+03	F =	.120811	729111	071682	E+03
H =	-.881309	686564	055228	E+02	H =	-.414027	955023	110336	E+04
I =	.395361	516315	538481	E+02	I =	.576824	678509	803280	E+01
J =	.167846	178671	487231	E+02	J =	.557071	095410	441769	E+03

N = 8

U =	.587785	252292	473129	E-00	U =	.951056	516295	153572	E-00
V =	.951056	516295	153572	E-00	V =	.587785	252292	473129	E-00
A =	-.220367	694788	998500	E+06	A =	.177572	769894	328699	E+07
B =	.469150	560807	870278	E+05	B =	-.611024	722362	626717	E+03
C =	-.114643	915868	034908	E+05	C =	-.462962	159216	725209	E+06
D =	-.558236	996816	845802	E+04	D =	-.490289	882116	709328	E+03
E =	.183543	158883	490441	E+04	E =	.103677	033309	869878	E+06
F =	.379541	240642	822960	E+03	F =	.379541	240642	822960	E+03
H =	-.387620	245158	602289	E+03	H =	-.182099	005463	397227	E+05
I =	.993651	868135	213498	E+02	I =	.144971	853793	255381	E+02
J =	.532592	098951	857671	E+02	J =	.192035	524144	749362	E+04

N = 9

U =	.587785	252292	473129	E-00	U =	.951056	516295	153572	E-00
V =	.951056	516295	153572	E-00	V =	.587785	252292	473129	E-00
A =	-.103783	619336	383772	E+07	A =	.110560	782973	484405	E+08
B =	.200889	328603	777023	E+06	B =	.181254	686365	721615	E+04
C =	-.566392	090358	438833	E+05	C =	-.254509	898078	734146	E+07
D =	-.206591	818907	776763	E+05	D =	-.199573	360264	144448	E+04
E =	.896397	062851	501168	E+04	E =	.494213	626120	057040	E+06
F =	.119236	397333	784846	E+04	F =	.119236	397333	784846	E+04
H =	-.156567	203302	235088	E+04	H =	-.735532	582872	621024	E+05
I =	.260137	450763	280057	E+03	I =	.379535	425711	820588	E+02
J =	.168426	211989	711298	E+03	J =	.638957	564136	845526	E+04

N - CUBE $\qquad W(X) = 1/\prod_{i=1}^{N} [1-(x^i)^2]^{\frac{1}{2}}$

N = 10

U =	.587785	252292	473129	E-00	U =	.951056	516295	153572	E-00
V =	.951056	516295	153572	E-00	V =	.587785	252292	473129	E-00
A =	-.460684	409827	191913	E+07	A =	.617321	530586	825619	E+08
B =	.825343	702815	861806	E+06	B =	.212346	463679	287028	E+05
C =	-.267329	307368	680100	E+06	C =	-.127310	196823	116432	E+08
D =	-.747096	849081	404085	E+05	D =	-.770059	687081	264990	E+04
E =	.403933	954186	221506	E+05	E =	.218468	421261	311042	E+07
F =	.374592	189904	332084	E+04	F =	.374592	189904	332084	E+04
H =	-.601174	903617	935006	E+04	H =	-.282424	237190	156354	E+06
I =	.700496	489349	854726	E+03	I =	.102201	060445	142595	E+03
J =	.531611	705886	524743	E+03	J =	.208736	562886	467371	E+05

N = 11

U =	.587785	252292	473129	E-00	U =	.951056	516295	153572	E-00
V =	.951056	516295	153572	E-00	V =	.587785	252292	473129	E-00
A =	-.195245	252659	718396	E+08	A =	.319141	897170	582023	E+09
B =	.328524	774451	221983	E+07	B =	.127518	863984	911087	E+06
C =	-.121170	331362	268179	E+07	C =	-.596296	876033	778936	E+08
D =	-.265516	842006	719376	E+06	D =	-.286871	759671	712203	E+05
E =	.172196	150673	967711	E+06	E =	.917172	346017	679764	E+07
F =	.117681	607189	556238	E+05	F =	.117681	607189	556238	E+05
H =	-.223203	696267	932146	E+05	H =	-.104858	225579	816855	E+07
I =	.192559	029670	607639	E+04	I =	.280939	838098	103085	E+03
J =	.167596	293977	267900	E+04	J =	.674619	812840	916112	E+05

N = 12

U =	.587785	252292	473129	E-00	U =	.951056	516295	153572	E-00
V =	.951056	516295	153572	E-00	V =	.587785	252292	473129	E-00
A =	-.797116	722394	239093	E+08	A =	.155713	747352	090707	E+10
B =	.127541	131177	860105	E+08	B =	.629304	630672	447670	E+06
C =	-.530405	384389	959464	E+07	C =	-.265987	436316	841196	E+09
D =	-.930936	485531	961004	E+06	D =	-.104244	997774	218082	E+06
E =	.704849	339667	424052	E+06	E =	.370792	748010	081126	E+08
F =	.369707	672609	349674	E+05	F =	.369707	672609	349674	E+05
H =	-.809094	337441	642134	E+05	H =	-.380102	112865	400736	E+07
I =	.537726	251551	608873	E+04	I =	.784532	027973	072500	E+03
J =	.527950	056931	552880	E+04	J =	.216545	102996	294275	E+06

N - CUBE $\quad W(X) = 1/\prod_{i=1}^{N}[1-(x^i)^2]^{\frac{1}{2}}$

N = 13

U =	.587785	252292	473129	E-00	U =	.951056	516295	153572	E-00
V =	.951056	516295	153572	E-00	V =	.587785	252292	473129	E-00
A =	-.315488	893269	137486	E+09	A =	.726085	263039	446232	E+10
B =	.485232	326338	009118	E+08	B =	.281378	151064	193867	E+07
C =	-.225368	842263	272287	E+08	C =	-.114241	843674	372207	E+10
D =	-.322870	025537	895362	E+07	D =	-.371859	560181	517797	E+06
E =	.279697	365693	962813	E+07	E =	.145639	008416	811419	E+09
F =	.116147	090824	531336	E+06	F =	.116147	090824	531336	E+06
H =	-.288075	747011	013266	E+06	H =	-.135334	280611	082157	E+08
I =	.152038	515736	522043	E+05	I =	.221821	205002	749606	E+04
J =	.166219	994005	781593	E+05	J =	.691875	254629	062990	E+06

N = 14

U =	.587785	252292	473129	E-00	U =	.951056	516295	153572	E-00
V =	.951056	516295	153572	E-00	V =	.587785	252292	473129	E-00
A =	-.121614	538388	747660	E+10	A =	.326409	304444	894127	E+11
B =	.181549	841164	132206	E+09	B =	.118401	878890	600800	E+08
C =	-.933546	886616	863696	E+08	C =	-.476037	015372	616986	E+10
D =	-.110985	471711	431176	E+08	D =	-.130760	563604	280264	E+07
E =	.108302	647351	971723	E+08	E =	.559122	990700	814813	E+09
F =	.364886	847270	174126	E+06	F =	.364886	847270	174126	E+06
H =	-.101148	919760	333124	E+07	H =	-.475184	614893	298275	E+08
I =	.434220	985545	958021	E+05	I =	.633519	880042	794231	E+04
J =	.523119	805096	702717	E+05	J =	.220335	216625	140614	E+07

N = 15

U =	.587785	252292	473129	E-00	U =	.951056	516295	153572	E-00
V =	.951056	516295	153572	E-00	V =	.587785	252292	473129	E-00
A =	-.458173	664826	524438	E+10	A =	.142370	102469	082740	E+12
B =	.669809	170853	876333	E+09	B =	.477908	042195	991473	E+08
C =	-.378353	751490	621970	E+09	C =	-.193499	139338	360761	E+11
D =	-.378682	342664	795434	E+08	D =	-.454582	176821	469421	E+07
E =	.411125	234864	518541	E+08	E =	.210710	209837	887250	E+10
F =	.114632	583877	551993	E+07	F =	.114632	583877	551993	E+07
H =	-.351218	040420	511910	E+07	H =	-.164997	717895	796311	E+09
I =	.125046	667004	022639	E+06	I =	.182440	628429	173530	E+05
J =	.164584	913018	130649	E+06	J =	.699995	157932	886895	E+07

N - CUBE $\quad W(X) = 1/\prod_{i=1}^{N} [1-(x^i)^2]^{\frac{1}{2}}$

N = 16

U =	.587785	252292	473129	E-00	U =	.951056	516295	153572	E-00
V =	.951056	516295	153572	E-00	V =	.587785	252292	473129	E-00
A =	-.169140	531644	690936	E+11	A =	.605400	661482	218181	E+12
B =	.244185	215373	923036	E+10	B =	.187088	694874	421396	E+09
C =	-.150476	679570	194629	E+10	C =	-.770393	426676	730467	E+11
D =	-.128394	863145	966847	E+09	D =	-.156566	902027	263993	E+08
E =	.153528	858854	910369	E+09	E =	.781972	461899	526086	E+10
F =	.360128	883371	733118	E+07	F =	.360128	883371	733118	E+07
H =	-.120846	820755	454834	E+08	H =	-.567722	820152	944836	E+09
I =	.362626	791152	978513	E+06	I =	.529065	358144	058532	E+05
J =	.517702	000258	462910	E+06	J =	.221981	198739	585491	E+08

N = 17

U =	.587785	252292	473129	E-00	U =	.951056	516295	153572	E-00
V =	.951056	516295	153572	E-00	V =	.587785	252292	473129	E-00
A =	-.613033	556880	169686	E+11	A =	.251913	210299	464150	E+13
B =	.881084	434677	001421	E+10	B =	.715360	554824	419001	E+09
C =	-.588727	395465	418023	E+10	C =	-.301372	432360	668405	E+12
D =	-.432984	226058	143313	E+09	D =	-.535084	233106	935375	E+08
E =	.565488	561952	925141	E+09	E =	.286466	979652	642085	E+11
F =	.113137	825434	613221	E+08	F =	.113137	825434	613221	E+08
H =	-.412664	656842	412452	E+08	H =	-.193864	547942	149012	E+10
I =	.105785	240143	239198	E+07	I =	.154338	585367	035380	E+06
J =	.162814	093123	279413	E+07	J =	.702951	909521	998964	E+08

N = 18

U =	.587785	252292	473129	E-00	U =	.951056	516295	153572	E-00
V =	.951056	516295	153572	E-00	V =	.587785	252292	473129	E-00
A =	-.218454	395373	255847	E+12	A =	.102877	991003	845667	E+14
B =	.315081	957996	423044	E+11	B =	.268445	507053	991042	E+10
C =	-.227047	175263	033820	E+11	C =	-.116124	247632	584794	E+13
D =	-.145331	362102	635041	E+10	D =	-.181678	000624	774206	E+09
E =	.205854	311086	139866	E+10	E =	.103789	392742	987495	E+12
F =	.355432	961228	505352	E+08	F =	.355432	961228	505352	E+08
H =	-.140013	819467	885804	E+09	H =	-.657766	914775	086534	E+10
I =	.310178	524406	083530	E+07	I =	.452544	368224	218212	E+06
J =	.511967	168382	041168	E+07	J =	.222357	423502	423013	E+09

N - CUBE $\qquad W(X) = 1/\prod_{i=1}^{N} [1-(x^i)^2]^{\frac{1}{2}}$

N = 19

U =	.587785	252292	473129	E-00	U =	.951056	516295	153572	E-00
V =	.951056	516295	153572	E-00	V =	.587785	252292	473129	E-00
A =	-.766143	464980	260129	E+12	A =	.413322	574800	722805	E+14
B =	.111791	848420	715841	E+12	B =	.992033	978175	022544	E+10
C =	-.864592	729356	672432	E+11	C =	-.441599	839009	170590	E+13
D =	-.485805	576725	100564	E+10	D =	-.613409	573960	771659	E+09
E =	.741822	481915	735073	E+10	E =	.372457	797442	117473	E+12
F =	.111662	557983	913822	E+09	F =	.111662	557983	913822	E+09
H =	-.472449	081948	124846	E+09	H =	-.221950	501887	863811	E+11
I =	.913551	162727	007255	E+07	I =	.133285	318372	184831	E+07
J =	.160968	869946	007497	E+08	J =	.702730	830026	844482	E+09

N = 20

U =	.587785	252292	473129	E-00	U =	.951056	516295	153572	E-00
V =	.951056	516295	153572	E-00	V =	.587785	252292	473129	E-00
A =	-.264604	574730	458996	E+13	A =	.163679	992436	975117	E+15
B =	.393884	542988	635648	E+12	B =	.361941	382912	599333	E+11
C =	-.325551	601354	507465	E+12	C =	-.166004	650150	851025	E+14
D =	-.161804	341080	057912	E+11	D =	-.206107	602784	622359	E+10
E =	.264977	981547	143605	E+11	E =	.132547	874152	939458	E+13
F =	.350798	271843	307978	E+09	F =	.350798	271843	307978	E+09
H =	-.158660	412125	336578	E+10	H =	-.745366	208687	444655	E+11
I =	.270118	176141	325151	E+08	I =	.394097	107792	407120	E+07
J =	.506057	981736	113002	E+08	J =	.221926	537257	720503	E+10

N = 21

U =	.587785	252292	473129	E-00	U =	.951056	516295	153572	E-00
V =	.951056	516295	153572	E-00	V =	.587785	252292	473129	E-00
A =	-.900174	695864	355025	E+13	A =	.639945	136813	179018	E+15
B =	.137920	223897	074790	E+13	B =	.130627	186018	708332	E+12
C =	-.121357	016231	720969	E+13	C =	-.617694	602304	911290	E+14
D =	-.537175	772700	113599	E+11	D =	-.689601	278430	987592	E+10
E =	.939187	883408	839759	E+11	E =	.468243	210793	962941	E+13
F =	.110206	527371	493156	E+10	F =	.110206	527371	493156	E+10
H =	-.530604	216448	388693	E+10	H =	-.249271	036063	665303	E+12
I =	.801456	762335	179700	E+08	I =	.116930	965760	596600	E+08
J =	.159083	156700	612075	E+09	J =	.700434	112244	644307	E+10

N - CUBF $\qquad W(X) = 1/\prod_{i=1}^{N} [1-(x^i)^2]^{\frac{1}{2}}$

N = 22

U =	.587785 252292 473129 E-00	U =	.951056 516295 153572 E-00
V =	.951056 516295 153572 E-00	V =	.587785 252292 473129 E-00
A =	-.301602 387182 177493 E+14	A =	.247353 929433 315599 E+16
B =	.480246 999102 501484 E+13	B =	.467059 544182 809221 E+12
C =	-.448325 397760 769357 E+13	C =	-.227756 891989 981283 E+15
D =	-.177823 008556 309170 E+12	D =	-.229869 211690 875413 E+11
E =	.330606 901815 948372 E+12	E =	.164335 599900 869978 E+14
F =	.346224 016767 925365 E+10	F =	.346224 016767 925365 E+10
H =	-.176796 911492 517957 E+11	H =	-.830569 149932 165350 E+12
I =	.238533 222005 247926 E+09	I =	.348015 280746 931513 E+08
J =	.500056 558318 161459 E+09	J =	.220956 161149 569676 E+11

N = 23

U =	.587785 252292 473129 E-00	U =	.951056 516295 153572 E-00
V =	.951056 516295 153572 E-00	V =	.587785 252292 473129 E-00
A =	-.994640 260149 353973 E+14	A =	.946286 319529 315483 E+16
B =	.166385 649854 539936 E+14	B =	.165646 201434 848607 E+13
C =	-.164281 394638 242767 E+14	C =	-.832952 751389 074441 E+15
D =	-.587123 677599 509859 E+12	D =	-.763701 672209 130092 E+11
E =	.115667 084265 450264 E+13	E =	.573395 569843 001561 E+14
F =	.108769 482757 446372 E+11	F =	.108769 482757 446372 E+11
H =	-.587162 385654 959833 E+11	H =	-.275841 336485 236504 E+13
I =	.711905 506994 350591 E+09	I =	.103865 613686 497198 E+09
J =	.157177 157782 389678 E+10	J =	.696722 396283 034768 E+11

N = 24

U =	.587785 252292 473129 E-00	U =	.951056 516295 153572 E-00
V =	.951056 516295 153572 E-00	V =	.587785 252292 473129 E-00
A =	-.322498 013527 066740 E+15	A =	.358658 121343 538504 E+17
B =	.573832 699147 917179 E+14	B =	.583297 055883 599556 E+13
C =	-.597559 131976 546358 E+14	C =	-.302389 812616 510867 E+16
D =	-.193396 411672 818518 E+13	D =	-.252976 094251 368149 E+12
E =	.402455 980291 106433 E+13	E =	.199019 170410 221840 E+15
F =	.341709 407965 555209 E+11	F =	.341709 407965 555209 E+11
H =	-.194433 449871 019658 E+12	H =	-.913423 338758 863753 E+13
I =	.213001 629602 244665 E+10	I =	.310765 189445 804837 E+09
J =	.494013 304351 770146 E+10	J =	.219611 763619 561435 E+12

N - CUBE $\qquad W(X) = 1/\prod_{i=1}^{N} [1-(x^i)^2]^{\frac{1}{2}}$

N = 25

U =	.587785	252292	473129	E-00	U =	.951056	516295	153572	E-00
V =	.951056	516295	153572	E-00	V =	.587785	252292	473129	E-00
A =	-.102613	026652	821634	E+16	A =	.134791	063834	954236	E+18
B =	.197083	128275	624081	E+15	B =	.204103	554606	875571	E+14
C =	-.215902	812931	714720	E+15	C =	-.109046	055682	353744	E+17
D =	-.635677	649041	918768	E+13	D =	-.835752	339952	421297	E+12
E =	.139338	206845	727210	E+14	E =	.687498	767041	518220	E+15
F =	.107351	176572	710582	E+12	F =	.107351	176572	710582	E+12
H =	-.642155	348892	397495	E+12	H =	-.301676	323274	755394	E+14
I =	.638747	793181	017615	E+10	I =	.931920	470874	636244	E+09
J =	.155263	602184	291247	E+11	J =	.692015	485374	436073	E+12

$$\text{N - CUBE} \qquad W(X) = 1/\prod_{i=1}^{N} [1-(x^i)^2]^{\frac{1}{2}}$$

ERRORS

		N = 4	
ER1 =	.13000E-17	ER1 =	.60000E-18
ER2 =	.50000E-17	ER2 =	.58100E-17
ER3 =	.30000E-17	ER3 =	.38000E-17
		N = 5	
ER1 =	-.10000E-17	ER1 =	.00000E-99
ER2 =	.15000E-16	ER2 =	.27700E-16
ER3 =	.11000E-16	ER3 =	.24600E-16
		N = 6	
ER1 =	-.20000E-17	ER1 =	.18000E-16
ER2 =	.60000E-16	ER2 =	.89700E-16
ER3 =	.40000E-16	ER3 =	.13800E-15
		N = 7	
ER1 =	-.12000E-15	ER1 =	.90000E-16
ER2 =	.60000E-16	ER2 =	.19500E-15
ER3 =	-.10000E-16	ER3 =	.13100E-15
		N = 8	
ER1 =	.16000E-15	ER1 =	.19000E-15
ER2 =	.12000E-14	ER2 =	.72000E-15
ER3 =	.12000E-14	ER3 =	.64000E-15
		N = 9	
ER1 =	-.12000E-14	ER1 =	.49000E-14
ER2 =	.29000E-14	ER2 =	.57200E-14
ER3 =	.39000E-14	ER3 =	.12150E-13
		N = 10	
ER1 =	-.50000E-15	ER1 =	-.20000E-15
ER2 =	.12000E-13	ER2 =	.93100E-14
ER3 =	.11000E-13	ER3 =	.17600E-13
		N = 11	
ER1 =	.80000E-14	ER1 =	-.10100E-12
ER2 =	.45000E-13	ER2 =	.25700E-13
ER3 =	.46000E-13	ER3 =	.40200E-13
		N = 12	
ER1 =	-.62000E-13	ER1 =	-.16200E-12
ER2 =	.54000E-12	ER2 =	.24310E-12
ER3 =	.62000E-12	ER3 =	.61300E-12
		N = 13	
ER1 =	.44000E-12	ER1 =	-.50000E-13
ER2 =	.42100E-11	ER2 =	.29240E-11
ER3 =	.47600E-11	ER3 =	.72590E-11
		N = 14	
ER1 =	-.24300E-11	ER1 =	-.54000E-11
ER2 =	.18000E-11	ER2 =	.12730E-11
ER3 =	.18000E-11	ER3 =	.27300E-11

N - CUBE $\quad W(X) = 1/\prod_{i=1}^{N} [1-(x^i)^2]^{\frac{1}{2}}$

ERRORS

N			
N = 15	ER1 = -.80000E-11	ER1 = -.80000E-11
	ER2 = .16800E-10	ER2 = .26280E-10
	ER3 = .21100E-10	ER3 = .63250E-10
N = 16	ER1 = -.60300E-10	ER1 = .30000E-10
	ER2 = .42200E-09	ER2 = .11736E-09
	ER3 = .46300E-09	ER3 = .32640E-09
N = 17	ER1 = -.16900E-09	ER1 = .32000E-10
	ER2 = .55000E-10	ER2 = .11290E-09
	ER3 = .73000E-10	ER3 = .14780E-09
N = 18	ER1 = -.67200E-09	ER1 = .13100E-09
	ER2 = .28000E-09	ER2 = .11169E-07
	ER3 = .43000E-09	ER3 = .31484E-07
N = 19	ER1 = -.10600E-08	ER1 = -.10600E-08
	ER2 = .10870E-07	ER2 = .16490E-08
	ER3 = .12070E-07	ER3 = .23820E-08
N = 20	ER1 = .22900E-08	ER1 = .52900E-08
	ER2 = .10150E-06	ER2 = .14598E-06
	ER3 = .11430E-06	ER3 = .41065E-06
N = 21	ER1 = -.25800E-07	ER1 = -.56000E-08
	ER2 = .28640E-06	ER2 = .26942E-06
	ER3 = .31640E-06	ER3 = .74617E-06
N = 22	ER1 = -.81000E-07	ER1 = .29000E-07
	ER2 = .80000E-08	ER2 = .14482E-05
	ER3 = .10000E-07	ER3 = .40594E-05
N = 23	ER1 = .75000E-07	ER1 = -.52400E-06
	ER2 = .67000E-07	ER2 = .31340E-06
	ER3 = .16500E-06	ER3 = .64280E-06
N = 24	ER1 = -.10600E-05	ER1 = -.60000E-07
	ER2 = .13970E-04	ER2 = .15206E-04
	ER3 = .15430E-04	ER3 = .42491E-04
N = 25	ER1 = .64000E-06	ER1 = .46500E-05
	ER2 = .13290E-04	ER2 = .26885E-04
	ER3 = .15100E-04	ER3 = .76004E-04

INFINITE N - SPACE $\quad W(X) = \exp\left[-\sum_{i=1}^{N} (x^i)^2\right]$

N = 4

U =	.958572	464613	818507	E-00	U =	.202018	287045	608563	E+01
V =	.202018	287045	608563	E+01	V =	.958572	464613	818507	E-00
A =	.434201	908835	033388	E-00	A =	-.129200	456295	440479	E+01
B =	.421269	636704	420208	E-00	B =	-.411070	968106	078771	E-01
C =	-.388835	089816	943788	E-01	C =	.128659	646042	805280	E+01
D =	.162248	779448	181111	E-00	D =	.416922	717921	280931	E-03
E =	-.694871	196535	468232	E-03	E =	-.270414	632413	635184	E-00
F =	.246740	110027	233965	E-01	F =	.246740	110027	233965	E-01
H =	.555896	957228	374581	E-03	H =	.216331	705930	908147	E-00
I =	.811243	897240	905554	E-01	I =	.208461	358960	640468	E-03
J =	-.694871	196535	468227	E-04	J =	-.270414	632413	635184	E-01

N = 5

U =	.958572	464613	818507	E-00	U =	.202018	287045	608563	E+01
V =	.202018	287045	608563	E+01	V =	.958572	464613	818507	E-00
A =	-.253668	010879	349665	E+01	A =	-.636120	674427	001381	E+01
B =	.123437	084608	348971	E+01	B =	-.158849	571125	812964	E-00
C =	-.154908	364315	628598	E-00	C =	.276812	270708	428094	E+01
D =	.250000	000000	000000	E-18	D =	-.108000	000000	000000	E-19
E =	-.147795	255381	913720	E-02	E =	-.575156	947875	296711	E-00
F =	.437335	458190	621571	E-01	F =	.437335	458190	621571	E-01
H =	.492650	851273	045730	E-03	H =	.191718	982625	098903	E-00
I =	.718946	184844	120889	E-01	I =	.184744	069227	392148	E-03
J =	.615813	564091	307162	E-04	J =	.239648	728281	373630	E-01

N = 6

U =	.958572	464613	818507	E-00	U =	.202018	287045	608563	E+01
V =	.202018	287045	608563	E+01	V =	.958572	464613	818507	E-00
A =	-.158476	525254	865747	E+02	A =	-.158476	525254	865747	E+02
B =	.441152	531950	349214	E+01	B =	-.430472	511168	700423	E-00
C =	-.430472	511168	700423	E-00	C =	.441152	531950	349214	E+01
D =	-.509719	573568	316405	E-00	D =	-.130980	134773	618583	E-02
E =	-.130980	134773	618583	E-02	E =	-.509719	573568	316405	E-00
F =	.775156	917007	495504	E-01	F =	.775156	917007	495504	E-01
H =	.901186	203032	846823	E-22	H =	.350703	752068	572091	E-19
I =	.849532	622613	860675	E-01	I =	.218300	224622	697636	E-03
J =	.218300	224622	697636	E-03	J =	.849532	622613	860675	E-01

INFINITE N - SPACE $\quad W(X) = \exp\left[-\sum_{i=1}^{N} (x^i)^2\right]$

N = 7

U = .958572 464613 818507 E-00	U = .202018 287045 608563 E+01
V = .202018 287045 608563 E+01	V = .958572 464613 818507 E-00
A = -.633366 882970 117412 E+02	A = -.212838 215618 614373 E+02
B = .141697 714894 558742 E+02	B = -.102075 384140 454166 E+01
C = -.105170 800730 742482 E+01	C = .212371 254208 147568 E+01
D = -.180690 884210 615978 E+01	D = -.464312 488543 247493 E-02
E = .309541 659028 831658 E-02	E = .120460 589473 743985 E+01
F = .137392 986260 598289 E-00	F = .137392 986260 598289 E-00
H = -.154770 829514 415830 E-02	H = -.602302 947368 719928 E-00
I = .112931 802631 634986 E-00	I = .290195 305339 529682 E-03
J = .483658 842232 549469 E-03	J = .188219 671052 724977 E-00

N = 8

U = .958572 464613 818507 E-00	U = .202018 287045 608563 E+01
V = .202018 287045 608563 E+01	V = .958572 464613 818507 E-00
A = -.211087 938411 669934 E+03	A = .274297 114645 976856 E+02
B = .406415 332648 986914 E+02	B = -.225513 590927 699030 E+01
C = -.240875 743483 416407 E+01	C = -.191415 007297 253874 E+02
D = -.480399 380313 943490 E+01	D = -.123445 868751 300346 E-01
E = .205743 114585 500576 E-01	E = .800665 633856 572484 E+01
F = .243522 727585 006093 E-00	F = .243522 727585 006093 E-00
H = -.548648 305561 334871 E-02	H = -.213510 835695 085995 E+01
I = .160133 126771 314497 E-00	I = .411486 229171 001153 E-03
J = .960134 534732 336025 E-03	J = .373643 962466 400492 E-00

N = 9

U = .958572 464613 818507 E-00	U = .202018 287045 608563 E+01
V = .202018 287045 608563 E+01	V = .958572 464613 818507 E-00
A = -.630610 850979 217708 E+03	A = .320602 584843 061566 E+03
B = .107123 596328 584155 E+03	B = -.476800 680810 685931 E+01
C = -.531258 093721 844488 E+01	C = -.104801 741316 589491 E+03
D = -.113531 430881 343025 E+02	D = -.291736 140595 492272 E-01
E = .729340 351488 730677 E-01	E = .283828 577203 357562 E+02
F = .431632 796291 058984 E-00	F = .431632 796291 058984 E-00
H = -.145868 070297 746135 E-01	H = -.567657 154406 715125 E+01
I = .236523 814336 131302 E-00	I = .607783 626240 608898 E-03
J = .182335 087872 182669 E-02	J = .709571 443008 393906 E-00

INFINITE N - SPACE $W(X) = \exp\left[-\sum_{i=1}^{N} (x^i)^2\right]$

N = 10

U =	.958572	464613	818507	E-00	U =	.202018	287045	608563	E+01
V =	.202018	287045	608563	E+01	V =	.958572	464613	818507	E-00
A =	-.174623	943790	206496	E+04	A =	.146515	469039	558422	E+04
B =	.265479	400792	893026	E+03	B =	-.978295	306119	119041	E+01
C =	-.114376	373679	009290	E+02	C =	-.378454	109173	346551	E+03
D =	-.251536	527330	562335	E+02	D =	-.646361	057308	491651	E-01
E =	.211144	612054	107272	E-00	E =	.821685	989279	836961	E+02
F =	.765049	211963	203633	E-00	F =	.765049	211963	203633	E-00
H =	-.344725	897231	195546	E-01	H =	-.134152	814576	299912	E+02
I =	.359337	896186	517621	E-00	I =	.923372	939012	130928	E-03
J =	.338570	077637	781340	E-02	J =	.131757	228601	723127	E+01

N = 11

U =	.958572	464613	818507	E-00	U =	.202018	287045	608563	E+01
V =	.202018	287045	608563	E+01	V =	.958572	464613	818507	E-00
A =	-.457017	801427	781881	E+04	A =	.521302	929575	996904	E+04
B =	.628339	236571	463413	E+03	B =	-.196394	294180	524062	E+02
C =	-.242220	099984	595940	E+02	C =	-.115500	830947	035974	E+04
D =	-.535004	263812	546948	E+02	D =	-.137477	417412	215635	E-00
E =	.549909	669648	862540	E-00	E =	.214001	705525	018779	E+03
F =	.135601	442187	641066	E+01	F =	.135601	442187	641066	E+01
H =	-.763763	430067	864640	E-01	H =	-.297224	591006	970527	E+02
I =	.557296	108138	069738	E-00	I =	.143205	643137	724619	E-02
J =	.620557	786930	140020	E-02	J =	.241494	980193	163553	E+01

N = 12

U =	.958572	464613	818507	E-00	U =	.202018	287045	608563	E+01
V =	.202018	287045	608563	E+01	V =	.958572	464613	818507	E-00
A =	-.114481	017286	102319	E+05	A =	.162963	518248	033227	E+05
B =	.143552	181353	500144	E+04	B =	-.387776	123011	517399	E+02
C =	-.506904	841098	321536	E+02	C =	-.320046	665050	927143	E+04
D =	-.110631	542891	965602	E+03	D =	-.284284	440888	964418	E-00
E =	.134019	807847	654654	E+01	E =	.521548	702204	980698	E+03
F =	.240347	298393	826109	E+01	F =	.240347	298393	826109	E+01
H =	-.162448	251936	551096	E-00	H =	-.632180	245096	946301	E+02
I =	.878028	118190	203197	E-00	I =	.225622	572134	098744	E-02
J =	.112811	286067	049372	E-01	J =	.439014	059095	101598	E+01

INFINITE N - SPACE $\quad W(X) = \exp\left[-\sum_{i=1}^{N} (x^i)^2\right]$

N = 13

U =	.958572 464613 818507 E-00	U =	.202018 287045 608563 E+01
V =	.202018 287045 608563 E+01	V =	.958572 464613 818507 E-00
A =	-.276893 741916 112750 E+05	A =	.468938 671804 479677 E+05
B =	.318950 709066 873661 E+04	B =	-.755720 146372 677731 E+02
C =	-.105133 036354 962875 E+03	C =	-.831439 875751 961122 E+04
D =	-.224102 061977 695088 E+03	D =	-.575864 059435 618885 E-00
E =	.311926 365527 626895 E+01	E =	.121388 616904 584839 E+04
F =	.426004 494592 874234 E+01	F =	.426004 494592 874234 E+01
H =	-.335920 701337 444349 E-00	H =	-.130726 202820 322134 E+03
I =	.140063 788736 059430 E+01	I =	.359915 037147 261802 E-02
J =	.203951 854383 448354 E-01	J =	.793694 802837 670103 E+01

N = 14

U =	.958572 464613 818507 E-00	U =	.202018 287045 608563 E+01
V =	.202018 287045 608563 E+01	V =	.958572 464613 818507 E-00
A =	-.650785 457777 489022 E+05	A =	.127205 604597 410381 E+06
B =	.692908 698813 912203 E+04	B =	-.145732 123976 726825 E+03
C =	-.216500 135212 362524 E+03	C =	-.206108 453624 049828 E+05
D =	-.446861 883091 761317 E+03	D =	-.114827 902846 283886 E+01
E =	.701726 072949 512636 E+01	E =	.273082 261889 409693 E+04
F =	.755073 306944 198016 E+01	F =	.755073 306944 198016 E+01
H =	-.680461 646496 497102 E-00	H =	-.264807 041832 154854 E+03
I =	.225687 819743 313796 E+01	I =	.579938 903264 060030 E-02
J =	.367294 638733 904686 E-01	J =	.142935 619170 765404 E+02

N = 15

U =	.958572 464613 818507 E-00	U =	.202018 287045 608563 E+01
V =	.202018 287045 608563 E+01	V =	.958572 464613 818507 E-00
A =	-.149342 555376 866388 E+06	A =	.329928 232936 401983 E+06
B =	.147775 209166 766739 E+05	B =	-.278587 399252 095220 E+03
C =	-.443218 256423 724827 E+03	C =	-.492898 817155 974051 E+05
D =	-.880046 739454 314274 E+03	D =	-.226141 287323 667042 E+01
E =	.153776 075380 093589 E+02	E =	.598431 782828 933706 E+04
F =	.133833 259060 920649 E+02	F =	.133833 259060 920649 E+02
H =	-.135684 772394 200225 E+01	H =	-.528028 043672 588564 E+03
I =	.366686 141439 297614 E+01	I =	.942255 363848 612677 E-02
J =	.659578 754694 028874 E-01	J =	.256680 299007 508329 E+02

INFINITE N - SPACE $\quad W(X) = \exp\left[-\sum_{i=1}^{N} (x^i)^2\right]$

N = 16

U =	.958572	464613	818507	E-00	U =	.202018	287045	608563	E+01
V =	.202018	287045	608563	E+01	V =	.958572	464613	818507	E-00
A =	-.335853	158762	564061	E+06	A =	.825836	214738	119576	E+06
B =	.310325	368617	976428	E+05	B =	-.528666	780504	093834	E+03
C =	-.902770	109698	620613	E+03	C =	-.114552	738154	982338	E+06
D =	-.171582	645555	490692	E+04	D =	-.440907	495122	120846	E+01
E =	.330012	579682	314694	E+02	E =	.128427	010461	230912	E+05
F =	.237213	275401	764350	E+02	F =	.237213	275401	764350	E+02
H =	-.267216	663710	376270	E+01	H =	-.103989	482154	842844	E+04
I =	.599939	320124	093331	E+01	I =	.154163	459832	909386	E-01
J =	.118191	985871	897196	E-00	J =	.459953	478761	804887	E+02

N = 17

U =	.958572	464613	818507	E-00	U =	.202018	287045	608563	E+01
V =	.202018	287045	608563	E+01	V =	.958572	464613	818507	E-00
A =	-.742323	844678	970251	E+06	A =	.200796	637158	128452	E+07
B =	.643197	940104	163593	E+05	B =	-.996972	298209	057275	E+03
C =	-.183055	969833	951926	E+04	C =	-.260077	348243	861723	E+06
D =	-.331769	804578	238948	E+04	D =	-.852532	568315	245220	E+01
E =	.696234	930790	783596	E+02	E =	.270945	340405	561807	E+05
F =	.420449	583471	767941	E+02	F =	.420449	583471	767941	E+02
H =	-.520992	125081	538745	E+01	H =	-.202748	213908	923801	E+04
I =	.987410	132673	330203	E+01	I =	.253729	931046	203934	E-01
J =	.211441	609205	169945	E-00	J =	.822841	777227	775169	E+02

N = 18

U =	.958572	464613	818507	E-00	U =	.202018	287045	608563	E+01
V =	.202018	287045	608563	E+01	V =	.958572	464613	818507	E-00
A =	-.161630	324214	228176	E+07	A =	.476520	401084	161313	E+07
B =	.131822	913711	899154	E+06	B =	-.186996	116095	438604	E+04
C =	-.369668	249595	572629	E+04	C =	-.579060	169065	757174	E+06
D =	-.637050	556717	109302	E+04	D =	-.163699	751987	712016	E+02
E =	.144811	319066	052937	E+03	E =	.563544	723249	750536	E+05
F =	.745227	483336	155291	E+02	F =	.745227	483336	155291	E+02
H =	-.100738	308915	515087	E+02	H =	-.392031	111825	913416	E+04
I =	.163346	296594	130590	E+02	I =	.419742	953814	646196	E-01
J =	.377768	658433	181576	E-00	J =	.147011	666934	717531	E+03

$$\text{INFINITE N - SPACE} \qquad W(X) = \exp\left[-\sum_{i=1}^{N} (x^i)^2\right]$$

N = 19

U =	.958572	464613	818507	E-00	U =	.202018	287045	608563	E+01
V =	.202018	287045	608563	E+01	V =	.958572	464613	818507	E-00
A =	-.347341	514252	683993	E+07	A =	.110776	212895	971081	E+08
B =	.267549	659928	384967	E+06	B =	-.349080	707354	956131	E+04
C =	-.743685	055246	897501	E+04	C =	-.126808	442903	726582	E+07
D =	-.121599	984420	356963	E+05	D =	-.312469	506249	274840	E+02
E =	.297590	005951	690323	E+03	E =	.115809	508971	768536	E+06
F =	.132088	132263	979472	E+03	F =	.132088	132263	979472	E+03
H =	-.193433	503868	598710	E+02	H =	-.752761	808316	495488	E+04
I =	.271428	536652	582508	E+02	I =	.697476	576449	274196	E-01
J =	.674227	357234	298389	E-00	J =	.262380	918764	163091	E+03

N = 20

U =	.958572	464613	818507	E-00	U =	.202018	287045	608563	E+01
V =	.202018	287045	608563	E+01	V =	.958572	464613	818507	E-00
A =	-.737852	550747	737638	E+07	A =	.252979	059697	157214	E+08
B =	.538410	293007	920486	E+06	B =	-.648938	334376	892825	E+04
C =	-.149077	175943	284286	E+05	C =	-.273765	118896	194880	E+07
D =	-.230925	386417	049057	E+05	D =	-.593397	621045	077567	E+02
E =	.605265	573465	979118	E+03	E =	.235543	894145	390038	E+06
F =	.234120	118690	207552	E+03	F =	.234120	118690	207552	E+03
H =	-.369225	186428	048263	E+02	H =	-.143686	907103	941635	E+05
I =	.452794	875327	547170	E+02	I =	.116352	474714	721091	E-00
J =	.120230	890538	545127	E+01	J =	.467888	037838	465409	E+03

N = 21

U =	.958572	464613	818507	E-00	U =	.202018	287045	608563	E+01
V =	.202018	287045	608563	E+01	V =	.958572	464613	818507	E-00
A =	-.155139	971287	749302	E+08	A =	.568828	363782	264772	E+08
B =	.107535	927436	868107	E+07	B =	-.120188	730064	385237	E+05
C =	-.297820	979595	504139	E+05	C =	-.583734	047506	075533	E+07
D =	-.436591	563121	859141	E+05	D =	-.112188	789177	548780	E+03
E =	.122005	308230	584298	E+04	E =	.474793	324895	021816	E+06
F =	.414967	105946	914853	E+03	F =	.414967	105946	914853	E+03
H =	-.701179	932359	679875	E+02	H =	-.272869	726951	161963	E+05
I =	.757971	463753	227676	E+02	I =	.194772	203433	244409	E-00
J =	.214249	423776	568850	E+01	J =	.833768	610128	550444	E+03

INFINITE N - SPACE $W(X) = \exp\left[-\sum_{i=1}^{N} (x^i)^2\right]$

N = 22

U =	.958572	464613	818507	E-00	U =	.202018	287045	608563	E+01
V =	.202018	287045	608563	E+01	V =	.958572	464613	818507	E-00
A =	-.323215	658993	546817	E+08	A =	.126164	947478	481280	E+09
B =	.213348	085941	478591	E+07	B =	-.221856	548732	658415	E+05
C =	-.593042	191357	515458	E+05	C =	-.123115	025573	782417	E+08
D =	-.822203	297161	210282	E+05	D =	-.211277	542119	059254	E+03
E =	.244004	847662	991961	E+04	E =	.949564	200054	809521	E+06
F =	.735510	044934	726492	E+03	F =	.735510	044934	726492	E+03
H =	-.132566	300937	448943	E+03	H =	-.515892	264885	465275	E+05
I =	.127276	052192	137814	E+03	I =	.327055	018760	153643	E-00
J =	.381564	188553	512584	E+01	J =	.148488	727557	494117	E+04

N = 23

U =	.958572	464613	818507	E-00	U =	.202018	287045	608563	E+01
V =	.202018	287045	608563	E+01	V =	.958572	464613	818507	E-00
A =	-.667848	291101	696787	E+08	A =	.276452	114960	690846	E+09
B =	.420751	181466	831028	E+07	B =	-.408289	537758	563800	E+05
C =	-.117722	117433	685654	E+06	C =	-.257160	721377	599994	E+08
D =	-.154304	195323	828182	E+06	D =	-.396507	910382	236518	E+03
E =	.484620	779356	066855	E+04	E =	.188594	016506	901111	E+07
F =	.130365	761152	424510	E+04	F =	.130365	761152	424510	E+04
H =	-.249653	128759	185955	E+03	H =	-.971544	933520	399665	E+05
I =	.214311	382394	205808	E+03	I =	.550705	431086	439608	E-00
J =	.679203	365006	608850	E+01	J =	.264317	371619	520497	E+04

N = 24

U =	.958572	464613	818507	E-00	U =	.202018	287045	608563	E+01
V =	.202018	287045	608563	E+01	V =	.958572	464613	818507	E-00
A =	-.136969	708424	851257	E+09	A =	.599211	103625	292161	E+09
B =	.825323	132650	187436	E+07	B =	-.749324	650027	046706	E+05
C =	-.232982	570856	827401	E+06	C =	-.532532	197835	308665	E+08
D =	-.288691	346613	295968	E+06	D =	-.741835	971153	244042	E+03
E =	.956577	962802	867317	E+04	E =	.372259	894317	144800	E+07
F =	.231067	295380	843546	E+04	F =	.231067	295380	843546	E+04
H =	-.468527	981780	996237	E+03	H =	-.182331	376808	397453	E+06
I =	.361768	604778	566375	E+03	I =	.929619	011470	230629	E-00
J =	.120850	471491	129981	E+02	J =	.470299	186212	136288	E+04

INFINITE N - SPACE $W(X) = \exp\left[-\sum_{i=1}^{N} (x^i)^2\right]$

N = 25

U =	.958572 464613 818507 E-00	U =	.202018 287045 608563 E+01
V =	.202018 287045 608563 E+01	V =	.958572 464613 818507 E-00
A =	-.279016 515365 883840 E+09	A =	.128614 028957 979202 E+10
B =	.161104 715208 381181 E+08	B =	-.137176 187906 549147 E+06
C =	-.459757 633748 132253 E+06	C =	-.109424 654320 657534 E+09
D =	-.538623 251608 247937 E+06	D =	-.138407 370927 452705 E+04
E =	.187772 666558 244170 E+05	E =	.730732 211348 523035 E+07
F =	.409556 117516 098499 E+04	F =	.409556 117516 098499 E+04
H =	-.876580 015873 867136 E+03	H =	-.341128 059351 890360 E+06
I =	.612071 876827 554474 E+03	I =	.157281 103326 650801 E+01
J =	.214950 841213 089429 E+02	J =	.836498 231664 324448 E+04

INFINITE N - SPACE $\quad W(X) = \exp\left[-\sum_{i=1}^{N} (x^i)^2\right]$

ERRORS

		N = 4			
ER1 =	.25000E-17		ER1 =	.27000E-17	
ER2 =	.41000E-16		ER2 =	.40600E-16	
ER3 =	.70000E-17		ER3 =	.69000E-17	
		N = 5			
ER1 =	.80000E-17		ER1 =	.90000E-17	
ER2 =	.70000E-16		ER2 =	.64000E-16	
ER3 =	.10000E-16		ER3 =	.93000E-17	
		N = 6			
ER1 =	.90000E-17		ER1 =	.90000E-17	
ER2 =	.10000E-15		ER2 =	.98900E-16	
ER3 =	.14000E-16		ER3 =	.14000E-16	
		N = 7			
ER1 =	.13000E-16		ER1 =	.13000E-16	
ER2 =	.19000E-15		ER2 =	.18400E-15	
ER3 =	.37000E-16		ER3 =	.31000E-16	
		N = 8			
ER1 =	.28000E-16		ER1 =	.28000E-16	
ER2 =	.42000E-15		ER2 =	.42700E-15	
ER3 =	.80000E-16		ER3 =	.85000E-16	
		N = 9			
ER1 =	.50000E-16		ER1 =	.50000E-16	
ER2 =	.60000E-15		ER2 =	.60100E-15	
ER3 =	.10000E-15		ER3 =	.13100E-15	
		N = 10			
ER1 =	.70000E-16		ER1 =	.50000E-16	
ER2 =	.11000E-14		ER2 =	.10840E-14	
ER3 =	.16000E-15		ER3 =	.16000E-15	
		N = 11			
ER1 =	.60000E-16		ER1 =	.60000E-16	
ER2 =	.19000E-14		ER2 =	.18700E-14	
ER3 =	.26000E-15		ER3 =	.36000E-15	
		N = 12			
ER1 =	.16000E-15		ER1 =	.16000E-15	
ER2 =	.41000E-14		ER2 =	.43800E-14	
ER3 =	.90000E-15		ER3 =	.11300E-14	
		N = 13			
ER1 =	.50000E-15		ER1 =	.60000E-15	
ER2 =	.15000E-13		ER2 =	.62700E-14	
ER3 =	.33000E-14		ER3 =	.83000E-15	
		N = 14			
ER1 =	.10000E-15		ER1 =	.21000E-14	
ER2 =	.13000E-13		ER2 =	.13720E-13	
ER3 =	.22000E-14		ER3 =	.33000E-14	

INFINITE N - SPACE $W(X) = \exp\left[-\sum_{i=1}^{N} (x^i)^2\right]$

ERRORS

N				
N = 15	ER1 =	-.12000E-14	ER1 =	-.20000E-15
	ER2 =	.22000E-13	ER2 =	.20500E-13
	ER3 =	.56000E-14	ER3 =	.39000E-14
N = 16	ER1 =	.18000E-14	ER1 =	-.20000E-15
	ER2 =	.65000E-13	ER2 =	.51900E-13
	ER3 =	.14000E-13	ER3 =	.19700E-13
N = 17	ER1 =	.11000E-13	ER1 =	.10000E-14
	ER2 =	.13000E-12	ER2 =	.22650E-12
	ER3 =	.29000E-13	ER3 =	.13580E-12
N = 18	ER1 =	-.18000E-13	ER1 =	.22000E-13
	ER2 =	.14000E-12	ER2 =	.28560E-12
	ER3 =	.47000E-13	ER3 =	.15500E-12
N = 19	ER1 =	.24000E-13	ER1 =	.14000E-13
	ER2 =	.18000E-12	ER2 =	.39100E-12
	ER3 =	.30000E-13	ER3 =	.11600E-12
N = 20	ER1 =	.43000E-13	ER1 =	.23000E-13
	ER2 =	.96000E-12	ER2 =	.47500E-12
	ER3 =	.21000E-12	ER3 =	.19700E-12
N = 21	ER1 =	.00000E-99	ER1 =	.00000E-99
	ER2 =	.33000E-11	ER2 =	.27020E-11
	ER3 =	.76000E-12	ER3 =	.15930E-11
N = 22	ER1 =	-.25000E-12	ER1 =	-.15000E-12
	ER2 =	.11000E-11	ER2 =	.38620E-11
	ER3 =	.31000E-12	ER3 =	.29200E-11
N = 23	ER1 =	-.61000E-12	ER1 =	-.10000E-13
	ER2 =	.73000E-11	ER2 =	.25100E-11
	ER3 =	.21500E-11	ER3 =	.75000E-12
N = 24	ER1 =	-.10400E-11	ER1 =	-.40000E-13
	ER2 =	.67000E-11	ER2 =	.48500E-11
	ER3 =	.22000E-11	ER3 =	.20000E-11
N = 25	ER1 =	.70000E-12	ER1 =	-.30000E-12
	ER2 =	.90000E-11	ER2 =	.16830E-10
	ER3 =	.32000E-11	ER3 =	.12480E-10